1 Generalized composite rheology

1.1 Making a composite viscous rheology

The stress in viscous materials can be described by the following constitutive relation:

$$\tau_{ij} = 2\eta_{ijkl}\dot{\varepsilon}_{kl} \tag{1}$$

where $\boldsymbol{\eta}$ is the viscosity of that element. If multiple elements experience the same stress, strain rates are summed:

$$\tau_{ij} = 2\eta_{\mathrm{a}ijkl}\dot{\varepsilon}_{\mathrm{a}kl} = 2\eta_{\mathrm{b}ijkl}\dot{\varepsilon}_{\mathrm{b}kl} \tag{2}$$

$$\dot{\varepsilon}_{kl} = \dot{\varepsilon}_{\mathrm{a}kl} + \dot{\varepsilon}_{\mathrm{b}kl} \tag{3}$$

If the elements are isotropic, the viscosities reduce to scalars, and an effective viscosity η_{eff} can be defined:

$$\tau_{ij} = 2\eta_{\mathrm{eff}}\dot{\varepsilon}_{ij} \tag{4}$$

$$\eta_{\mathrm{eff}} = \left(\eta_{\mathrm{a}}^{-1} + \eta_{\mathrm{b}}^{-1}\right)^{-1} \tag{5}$$

If multiple elements experience the same strain rate, stresses are summed.

$$\tau_{ij} = 2\eta_{\mathrm{eff}ijkl}\dot{\varepsilon}_{kl} \tag{6}$$

$$\eta_{\mathrm{eff}ijkl} = \eta_{\mathrm{a}ijkl} + \eta_{\mathrm{b}ijkl} \tag{7}$$

The reduction to isotropic elements is trivial:

$$\eta_{\mathrm{eff}} = \eta_{\mathrm{a}} + \eta_{\mathrm{b}} \tag{8}$$

1.2 Modelling firmoviscous (Kelvin) elements as viscous elements

Incompressible elastic elements with infinite tensile strength (infinite Young's modulus) can only deform by shear. If the material is elastically isotropic, resistance to shear deformation is given by the scalar shear modulus G. For small elastic strains $\boldsymbol{\varepsilon}_{\mathrm{el}}$,

$$\tau_{\mathrm{el}ij} = 2G\varepsilon_{\mathrm{el}} \tag{9}$$

We can rearrange and differentiate to obtain an expression for the strain rate:

$$\dot{\varepsilon}_{\mathrm{el}ij} = \frac{\tilde{\dot{\tau}}_{ij}}{2G} = \frac{\dot{\tau}_{\mathrm{el}ij} + \tau_{\mathrm{el}ik}W_{kj} - W_{ik}\tau_{\mathrm{el}kj}}{2G} \tag{10}$$

where $\tilde{\dot{\boldsymbol{\tau}}}$ is the objective co-rotational (Jaumann) stress rate tensor and $W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ the material spin tensor. Discretizing the rate of deviatoric stress over an elastic timestep Δt_{el}:

$$\dot{\varepsilon}_{\mathrm{el}ij} = \frac{\left(\tau_{\mathrm{el}ij}^{t+\Delta t_{\mathrm{el}}} - \tau_{\mathrm{el}ij}^{t}\right)/\Delta t_{\mathrm{el}} + \tau_{\mathrm{el}ik}^{t}W_{kj}^{t} - W_{ik}^{t}\tau_{\mathrm{el}kj}^{t}}{2G} \tag{11}$$

Grouping the terms from the current timestep into a single parameter $\boldsymbol{\tau}_{\mathrm{el}}^{0}$ and renaming the stress for the following timestep:

$$\tau_{ij}^{0} = \tau_{ij}^{t} + \Delta t_{\mathrm{el}}\left(W_{ik}^{t}\tau_{kj}^{t} - \tau_{ik}^{t}W_{kj}^{t}\right) \tag{12}$$

$$\tau_{\mathrm{el}ij} = \tau_{ij}^{t+\Delta t_{\mathrm{el}}} \tag{13}$$

$$\dot{\varepsilon}_{\mathrm{el}ij} = \frac{\left(\tau_{\mathrm{el}ij} - \tau_{ij}^{0}\right)}{2\Delta t_{\mathrm{el}}G} \tag{14}$$

Consider now an isotropic viscous damper in parallel with the elastic element to make an Kelvin element:

$$\tau_{\mathrm{kelvin}ij} = \tau_{\mathrm{el}ij} + \tau_{\mathrm{d}ij} \tag{15}$$

$$= 2\left(\Delta t_{\mathrm{el}}G + \eta_{\mathrm{d}}\right)\dot{\varepsilon}_{\mathrm{kelvin}ij} + \tau_{\mathrm{el}ij}^{0} \tag{16}$$

We can find the damped elastic strain rate by rearranging:

$$\dot{\varepsilon}_{\mathrm{kelvin}ij} = \frac{\tau_{\mathrm{kelvin}ij} - \tau_{\mathrm{el}ij}^{0}}{2\left(\Delta t_{\mathrm{el}}G + \eta_{\mathrm{d}}\right)} \tag{17}$$

The tensors $\boldsymbol{\tau}_{\mathrm{kelvin}}$ and $\boldsymbol{\tau}_{\mathrm{el}}^{0}$ may not be scaled versions of each other. For this reason, the second invariant of the strain rate must be calculated from the sum of the two tensors, rather than by adding the two scalar strain rate invariants together as is usual for isotropic viscous deformation.

1.3 A composite elastoviscoplastic rheology

An example of an elastoviscoplastic rheology is shown in Figure 1. In this example, three viscous creep mechanisms (diffusion, dislocation and Peierls creep) act simultaneously with a pseudoplastic element (i.e. a viscous element with a very high stress exponent). In order to limit the effective viscosity between absolute limits ($\eta_{\rm min} < \eta < \eta_{\rm max}$), this set of elements are arranged in parallel (isostrain) with a constant, isotropic viscosity damper of viscosity $\eta_{\rm limiter}$ ($\eta_{\rm limiter} = (\eta_{\rm min}^{-1} - \eta_{\rm max}^{-1})^{-1}$, and then the entire package of components is arranged in series with another constant, isotropic viscosity damper of viscosity $\eta_{\rm max}$. A Kelvin element is then added in series. Separating the elastic element from the viscoplastic elements is done for computational convenience only (with this arrangement, the stored elastic stress can be easily separated from the other components). In the following, we use the tensor identity:

$$M_{ij} = ||\boldsymbol{M}||\, M^*_{ij}, \tag{18}$$

where $||\boldsymbol{M}||$ is the norm of $\boldsymbol{M}$ ($||\boldsymbol{M}|| = \sqrt{\boldsymbol{M}_{\rm II}} = \sqrt{\frac{1}{2}((M_{kk})^2 - M_{ij}M_{ji})}$) and $\boldsymbol{M}^*$ is the normalised tensor.

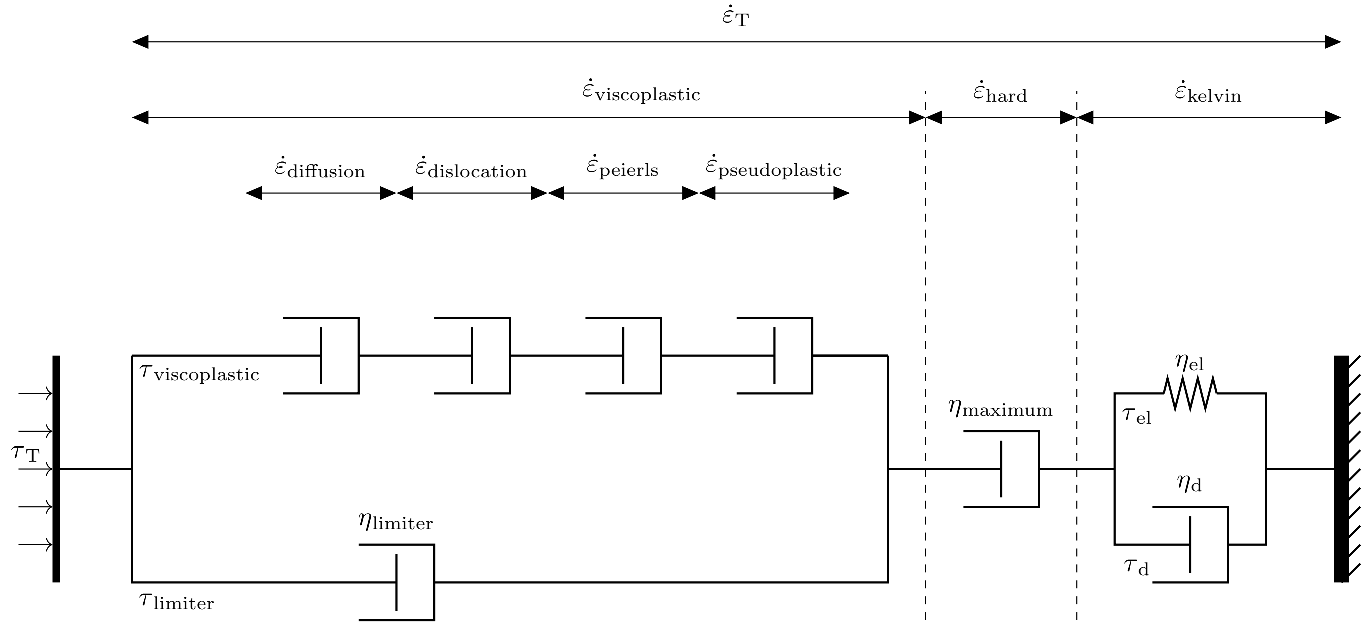

Figure 1: Proposal for a generalized elastoviscoplastic model in ASPECT.

The total shear stress is given by:

$$\tau_{{\rm T}ij} = \tau_{{\rm viscoplastic}ij} + \tau_{{\rm limiter}ij} = \tau_{{\rm hard}ij} = \tau_{{\rm kelvin}ij} \tag{19}$$

The strain rate is equal to the sum of the viscoplastic, hard and kelvin elements:

$$\dot\varepsilon_{{\rm T}ij} = \dot\varepsilon_{{\rm viscoplastic}ij} + \dot\varepsilon_{{\rm hard}ij} + \dot\varepsilon_{{\rm kelvin}ij} \tag{20}$$

$$= ||\dot{\boldsymbol{\varepsilon}}_{\rm viscoplastic}||\tau^*_{{\rm viscoplastic}ij} + \frac{||\boldsymbol{\tau}_{\rm T}||}{2\eta_{\rm max}}\tau^*_{{\rm T}ij} + \frac{||\boldsymbol{\tau}_{\rm T}||}{2\eta_{\rm kelvin}}\tau^*_{{\rm T}ij} - \frac{\tau^0_{{\rm el}ij}}{2\eta_{\rm kelvin}} \tag{21}$$

$$\dot\varepsilon_{{\rm T}ij} + \frac{\tau^0_{{\rm el}ij}}{2\eta_{\rm kelvin}} = ||\dot{\boldsymbol{\varepsilon}}_{\rm viscoplastic}||\tau^*_{{\rm viscoplastic}ij} + \frac{||\boldsymbol{\tau}_{\rm T}||}{2\eta_{\rm mk}}\tau^*_{{\rm T}ij} \tag{22}$$

$$\eta_{\rm mk} = \left(\eta_{\rm max}^{-1} + \eta_{\rm kelvin}^{-1}\right)^{-1} \tag{23}$$

Now we need to demonstrate that $\tau^*_{{\rm T}ij} = \tau^*_{{\rm viscoplastic}ij}$:

$$\tau_{{\rm T}ij} = ||\boldsymbol{\tau}_{\rm T}||\tau^*_{{\rm T}ij} = \tau_{{\rm viscoplastic}ij} + \tau_{{\rm limiter}ij} \tag{24}$$

$$= \tau_{{\rm viscoplastic}ij} + 2\eta_{\rm limiter}\dot\varepsilon_{{\rm viscoplastic}ij} \tag{25}$$

$$= ||\boldsymbol{\tau}_{\rm viscoplastic}||\tau^*_{{\rm viscoplastic}ij} + 2\eta_{\rm limiter}||\dot{\boldsymbol{\varepsilon}}_{\rm viscoplastic}||\tau^*_{{\rm viscoplastic}ij} \tag{26}$$

Therefore

$$\tau^*_{{\rm T}ij} = \tau^*_{{\rm viscoplastic}ij} \tag{27}$$

$$||\boldsymbol{\tau}_{\rm T}|| = ||\boldsymbol{\tau}_{\rm viscoplastic}|| + 2\eta_{\rm limiter}||\dot{\boldsymbol{\varepsilon}}_{\rm viscoplastic}|| \tag{28}$$

Substituting these expressions, we have

$$\dot{\varepsilon}_{\mathrm{T}ij} + \frac{\tau^0_{\mathrm{el}ij}}{2\eta_{\mathrm{kelvin}}} = ||\dot{\boldsymbol{\varepsilon}}_{\mathrm{viscoplastic}}||\tau^*_{\mathrm{viscoplastic}ij} + \frac{||\boldsymbol{\tau}_{\mathrm{viscoplastic}}|| + 2\eta_{\mathrm{limiter}}||\dot{\boldsymbol{\varepsilon}}_{\mathrm{viscoplastic}}||}{2\eta_{\mathrm{mk}}}\tau^*_{\mathrm{viscoplastic}ij} \tag{29}$$

(30)

1.4 The effective strain rate and viscoplastic stress derivatives

Defining the "effective" strain rate as

$$\dot{\varepsilon}_{\mathrm{eff}ij} = \dot{\varepsilon}_{\mathrm{T}ij} + \frac{\tau^0_{\mathrm{el}ij}}{2\eta_{\mathrm{kelvin}}} = \left|\left|\dot{\boldsymbol{\varepsilon}}_{\mathrm{T}} + \frac{\boldsymbol{\tau}^0_{\mathrm{el}}}{2\eta_{\mathrm{kelvin}}}\right|\right| \tau^*_{\mathrm{viscoplastic}ij} \tag{31}$$

we obtain the scalar equality

$$||\dot{\boldsymbol{\varepsilon}}_{\mathrm{eff}}|| = \left(1 + \frac{\eta_{\mathrm{limiter}}}{\eta_{\mathrm{mk}}}\right)||\dot{\boldsymbol{\varepsilon}}_{\mathrm{viscoplastic}}|| + \frac{||\boldsymbol{\tau}_{\mathrm{viscoplastic}}||}{2\eta_{\mathrm{mk}}} \tag{32}$$

$$= \left(\left(\frac{\eta_{\mathrm{max}}}{\eta_{\mathrm{max}} - \eta_{\mathrm{min}}}\right)\left(1 + \frac{\eta_{\mathrm{min}}}{\eta_{\mathrm{kelvin}}}\right)\right)||\dot{\boldsymbol{\varepsilon}}_{\mathrm{viscoplastic}}|| + \frac{1}{2}\left(\frac{1}{\eta_{\mathrm{max}}} + \frac{1}{\eta_{\mathrm{kelvin}}}\right)||\boldsymbol{\tau}_{\mathrm{viscoplastic}}|| \tag{33}$$

The derivative with respect to the viscoplastic (not total) stress is then

$$\frac{\partial\,||\dot{\boldsymbol{\varepsilon}}_{\mathrm{eff}}||}{\partial||\boldsymbol{\tau}_{\mathrm{viscoplastic}}||} = \left(\left(\frac{\eta_{\mathrm{max}}}{\eta_{\mathrm{max}} - \eta_{\mathrm{min}}}\right)\left(1 + \frac{\eta_{\mathrm{min}}}{\eta_{\mathrm{kelvin}}}\right)\right)\frac{\partial||\dot{\boldsymbol{\varepsilon}}_{\mathrm{viscoplastic}}||}{\partial||\boldsymbol{\tau}_{\mathrm{viscoplastic}}||} + \frac{1}{2}\left(\frac{1}{\eta_{\mathrm{max}}} + \frac{1}{\eta_{\mathrm{kelvin}}}\right) \tag{34}$$

$$\frac{\partial \ln||\dot{\boldsymbol{\varepsilon}}_{\mathrm{eff}}||}{\partial \ln||\boldsymbol{\tau}_{\mathrm{viscoplastic}}||} = \left[\left(\left(\frac{\eta_{\mathrm{max}}}{\eta_{\mathrm{max}} - \eta_{\mathrm{min}}}\right)\left(1 + \frac{\eta_{\mathrm{min}}}{\eta_{\mathrm{kelvin}}}\right)\right)\frac{\partial \ln||\dot{\boldsymbol{\varepsilon}}_{\mathrm{viscoplastic}}||}{\partial \ln||\boldsymbol{\tau}_{\mathrm{viscoplastic}}||} + \frac{1}{2}\left(\frac{1}{\eta_{\mathrm{max}}} + \frac{1}{\eta_{\mathrm{kelvin}}}\right)||\boldsymbol{\tau}_{\mathrm{viscoplastic}}||\right]||\dot{\boldsymbol{\varepsilon}}_{\mathrm{eff}}||^{-1} \tag{35}$$

1.5 The effective viscosity, body force and stored stress

$$\dot{\varepsilon}_{\mathrm{eff}ij} = \frac{\tau_{\mathrm{viscoplastic}ij}}{2\eta_{\mathrm{viscoplastic}}} + \frac{\tau_{\mathrm{T}ij}}{2\eta_{\mathrm{max}}} + \frac{\tau_{\mathrm{T}ij}}{2\eta_{\mathrm{kelvin}}} \tag{36}$$

The total stress is related to the viscoplastic stress via:

$$\tau_{\mathrm{T}ij} = \tau_{\mathrm{viscoplastic}ij} + \tau_{\mathrm{limiter}ij} \tag{37}$$

$$= \tau_{\mathrm{viscoplastic}ij} + 2\eta_{\mathrm{limiter}}\dot{\varepsilon}_{\mathrm{viscoplastic}ij} \tag{38}$$

$$= \left(1 + \frac{\eta_{\mathrm{limiter}}}{\eta_{\mathrm{viscoplastic}}}\right)\tau_{\mathrm{viscoplastic}ij} \tag{39}$$

Such that the effective strain rate is

$$\dot{\varepsilon}_{\mathrm{eff}ij} = \frac{\tau_{\mathrm{T}ij}}{2(\eta_{\mathrm{viscoplastic}} + \eta_{\mathrm{limiter}})} + \frac{\tau_{\mathrm{T}ij}}{2\eta_{\mathrm{max}}} + \frac{\tau_{\mathrm{T}ij}}{2\eta_{\mathrm{kelvin}}} \tag{40}$$

$$= \frac{\tau_{\mathrm{T}ij}}{2\eta_{\mathrm{eff}}} \tag{41}$$

with the effective viscosity

$$\eta_{\mathrm{eff}} = \left((\eta_{\mathrm{viscoplastic}} + \eta_{\mathrm{limiter}})^{-1} + \eta_{\mathrm{max}}^{-1} + \eta_{\mathrm{kelvin}}^{-1}\right)^{-1} \tag{42}$$

The total stress can therefore be written:

$$\tau_{\mathrm{T}ij} = 2\eta_{\mathrm{eff}}\dot{\varepsilon}_{\mathrm{eff}ij} = 2\eta_{\mathrm{eff}}\dot{\varepsilon}_{\mathrm{T}ij} + \frac{\eta_{\mathrm{eff}}}{\eta_{\mathrm{kelvin}}}\tau^0_{\mathrm{el}ij} \tag{43}$$

allowing the momentum equation to be expressed as the sum of viscous and body force parts:

$$-\partial_j(2\eta_{\mathrm{eff}}\dot{\varepsilon}_{\mathrm{T}ji}) + \partial_i(p) = \rho g_i + \partial_j F_{ji} \tag{44}$$

$$F_{ij} = \frac{\eta_{\mathrm{eff}}}{\eta_{\mathrm{kelvin}}}\tau^0_{\mathrm{el}ij} \tag{45}$$

The stored stress is the elastic stress:

$$\begin{aligned}
\tau_{\mathrm{el}ij} &= \tau_{\mathrm{T}ij} - 2\eta_d \dot{\varepsilon}_{\mathrm{kelvin}ij} && (46)\\
&= \tau_{\mathrm{T}ij} - \frac{\eta_{\mathrm{d}}}{\eta_{\mathrm{kelvin}}}(\tau_{\mathrm{T}ij} - \tau^0_{\mathrm{el}ij}) && (47)\\
&= \left(\frac{\eta_{\mathrm{el}}}{\eta_{\mathrm{el}}+\eta_{\mathrm{d}}}\right)\tau_{\mathrm{T}ij} + \left(\frac{\eta_{\mathrm{d}}}{\eta_{\mathrm{el}}+\eta_{\mathrm{d}}}\right)\tau^0_{\mathrm{el}ij} && (48)\\
&= 2\eta_{\mathrm{eff}}\left(\frac{\eta_{\mathrm{el}}}{\eta_{\mathrm{el}}+\eta_{\mathrm{d}}}\right)\dot{\varepsilon}_{\mathrm{eff}ij} + \left(\frac{\eta_{\mathrm{d}}}{\eta_{\mathrm{el}}+\eta_{\mathrm{d}}}\right)\tau^0_{\mathrm{el}ij} && (49)
\end{aligned}$$

so that the change in the stored elastic stress is

$$\Delta\tau_{\mathrm{el}ij} = \left(\frac{\eta_{\mathrm{el}}}{\eta_{\mathrm{el}}+\eta_{\mathrm{d}}}\right)\left(\tau_{\mathrm{viscoplastic}ij} - \tau^0_{\mathrm{el}ij}\right) \quad (50)$$

1.6 Multiple components (isostress)

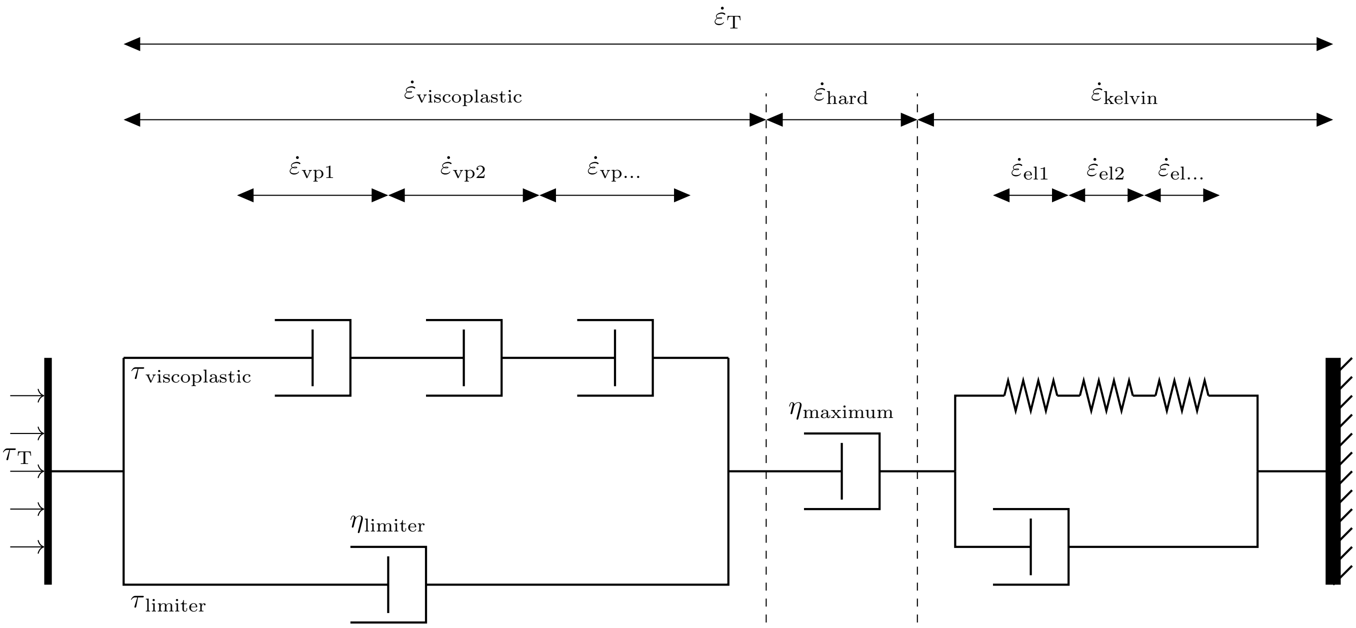

Figure 2: Proposal for a generalized elastoviscoplastic model in ASPECT.

For viscous elements arranged in parallel

$$\begin{aligned}
\dot{\varepsilon}_{\mathrm{viscoplastic}ij}(\boldsymbol{\tau}_{\mathrm{viscoplastic}}) &= \sum_k f_k \dot{\varepsilon}_{\mathrm{viscoplastic}ijk}(\boldsymbol{\tau}_{\mathrm{viscoplastic}}) && (51)\\
&= \sum_k \frac{f_k}{2\eta_{\mathrm{viscoplastic}k}}\tau_{\mathrm{viscoplastic}ij} && (52)
\end{aligned}$$

Such that the effective viscosity for a composite where each deformation mechanism is isotropic is

$$\eta_{\mathrm{viscoplastic}} = \left(\sum_k \frac{f_k}{\eta_{\mathrm{viscoplastic}k}}\right)^{-1} \quad (53)$$

A similar statement is true of effective elastic moduli

$$G = \left(\sum_k \frac{f_k}{G_k}\right)^{-1} \quad (54)$$