Consistent Linearization for Compressible Stokes System with Plastic Dilation

Yimin Jin

June 30, 2024

1 The Compressible Stokes System

The compressible Stokes system (without thermal expansion) is given by

$$-\nabla \cdot \boldsymbol{\tau} + \nabla p = \boldsymbol{f}, \tag{1}$$
$$-\nabla \cdot \boldsymbol{u} = \beta \dot{p}, \tag{2}$$

where $\boldsymbol{\tau}$ is the deviatoric stress tensor, p is pressure, $\beta := \frac{1}{\rho}\frac{\partial \rho}{\partial p}$ characterizes the compressiblity of the material, and $\boldsymbol{f}$ represents a body force. Approximating the time derivative of p with a backward Euler scheme, we can rewrite Eq. (2) as

$$\nabla \cdot \boldsymbol{u} + \frac{\beta p}{\Delta t} = \frac{\beta p^0}{\Delta t}, \tag{3}$$

where Δt is the time step length, and p^0 denotes the pressure in the previous time step.

We derive the weak for of the momentum conservative equation by integrating the inner product of Eq. (1) and a virtual velocity $\boldsymbol{v}$ across the computational domain Ω, which yields after integrating by part

$$\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\tau} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} p \mathrm{d}\Omega = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \mathrm{d}\Omega, \tag{4}$$

where $\boldsymbol{\varepsilon}(\cdot) := \frac{1}{2}[\nabla(\cdot) + \nabla^T(\cdot)] - \frac{1}{3}\nabla \cdot (\cdot)$ is the deviatoric symmetric gradient operator. Similarly, the weak form of mass conservative equation is obtained by multiplying Eq. (3) by a virtual pressure q and integrating across Ω, which gives

$$\int_\Omega q \nabla \cdot \boldsymbol{u} \mathrm{d}\Omega + \int_\Omega q \frac{\beta p}{\Delta t} \mathrm{d}\Omega = \int_\Omega q \frac{\beta p^0}{\Delta t} \mathrm{d}\Omega. \tag{5}$$

2 Constitutive Relation

Here we consider a Maxwell-type viscoelastic plastic model, which is based on the additive decomposition of the deviatoric strain rate $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^v + \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p. \tag{6}$$

The constitutive relationship between $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$ can then be expressed as

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{\tau}}{2\eta} + \frac{\mathring{\boldsymbol{\tau}}}{2G} + \gamma \frac{\partial \Psi}{\partial \boldsymbol{\tau}}, \tag{7}$$

where $\eta = \eta(\boldsymbol{u}, p)$ is the viscosity, G is the shear modulus, γ is the plastic multiplyer, Ψ is the plastic potential, and $\mathring{\boldsymbol{\tau}}$ denotes the co-rotational derivative of $\boldsymbol{\tau}$. We assume that the plastic flow is governed by the Drucker-Prager model:

$$\Phi = \tau_{\mathrm{II}} - \xi p - \zeta, \tag{8}$$

$$\Psi = \tau_{\mathrm{II}} - \bar{\xi} p, \tag{9}$$

where $\tau_{\mathrm{II}} := \sqrt{\frac{1}{2}\boldsymbol{\tau} : \boldsymbol{\tau}}$ stands for the second invariant of $\boldsymbol{\tau}$, ξ, $\bar{\xi}$ and ζ are material parameters related with frictional angle ϕ, dilatancy angle ψ and cohesion c. Integrating the stress rate with a first-order difference scheme, i.e. $\boldsymbol{\tau} = \boldsymbol{\tau}^0 + \mathring{\boldsymbol{\tau}}\Delta t$, we can rewrite Eq. (7) as

$$\boldsymbol{\tau} = 2\eta^{\mathrm{ve}} \left(\tilde{\boldsymbol{\varepsilon}} - \gamma \frac{\partial \Psi}{\partial \boldsymbol{\tau}} \right) = 2\eta^{\mathrm{ve}} \left(\tilde{\boldsymbol{\varepsilon}} - \gamma \frac{\boldsymbol{\tau}}{2\tau_{\mathrm{II}}} \right), \tag{10}$$

where η^{ve} and $\tilde{\boldsymbol{\varepsilon}}$ are defined as

$$\eta^{\mathrm{ve}} := \left(\frac{1}{\eta(\boldsymbol{u}, p)} + \frac{1}{G\Delta t} \right)^{-1}, \qquad \tilde{\boldsymbol{\varepsilon}} := \boldsymbol{\varepsilon} + \frac{\boldsymbol{\tau}^0}{2G\Delta t}. \tag{11}$$

The volumetric constitutive relation is based on an additive decomposition of the divergence of velocity (notice the negative sign for the plastic component)

$$\nabla \cdot \boldsymbol{u} = -\frac{\beta(p - p^0)}{\Delta t} - \gamma \frac{\partial \Psi}{\partial p} = -\frac{\beta(p - p^0)}{\Delta t} + \gamma \bar{\xi}. \tag{12}$$

3 Newton Linearization

Substituting Eq. (12) in Eqs. (4) and (5), we obtain the following nonlinear system

$$F_u := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\tau} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} p \mathrm{d}\Omega - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \mathrm{d}\Omega = 0, \tag{13}$$

$$F_p := -\int_\Omega q \nabla \cdot \boldsymbol{u} \mathrm{d}\Omega - \int_\Omega \frac{q\beta p}{\Delta t} \mathrm{d}\Omega + \int_\Omega q \left(\frac{\beta p^0}{\Delta t} + \gamma \bar{\xi} \right) \mathrm{d}\Omega = 0. \tag{14}$$

If we apply the Newton-Raphson method to solve Eqs. (13) and (14), then in each iteration we need to solve a linear equation set

$$\frac{\partial F_u}{\partial \boldsymbol{\varepsilon}} : \mathrm{d}\boldsymbol{\varepsilon} + \frac{\partial F_u}{\partial p} : \mathrm{d}p = -F_u, \tag{15}$$

$$\frac{\partial F_p}{\partial \boldsymbol{\varepsilon}} : \mathrm{d}\boldsymbol{\varepsilon} + \frac{\partial F_p}{\partial p} : \mathrm{d}p = -F_p. \tag{16}$$

The differentiations of F_u and F_p are given by

$$\frac{\partial F_u}{\partial \boldsymbol{\varepsilon}} : \mathrm{d}\boldsymbol{\varepsilon} = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \frac{\partial \boldsymbol{\tau}}{\partial \tilde{\boldsymbol{\varepsilon}}} : \mathrm{d}\tilde{\boldsymbol{\varepsilon}}\mathrm{d}\Omega, \tag{17}$$

$$\frac{\partial F_u}{\partial p}\mathrm{d}p = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \frac{\partial \boldsymbol{\tau}}{\partial p}\mathrm{d}p\mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v}\mathrm{d}p\mathrm{d}\Omega, \tag{18}$$

$$\frac{\partial F_p}{\partial \boldsymbol{\varepsilon}} : \mathrm{d}\boldsymbol{\varepsilon} = -\int_\Omega q\nabla \cdot \mathrm{d}\boldsymbol{u}\mathrm{d}\Omega + \int_\Omega q\bar{\xi}\frac{\partial \gamma}{\partial \tilde{\boldsymbol{\varepsilon}}} : \mathrm{d}\tilde{\boldsymbol{\varepsilon}}\mathrm{d}\Omega, \tag{19}$$

$$\frac{\partial F_p}{\partial p}\mathrm{d}p = -\int_\Omega \frac{q\beta\mathrm{d}p}{\Delta t}\mathrm{d}\Omega + \int_\Omega q\bar{\xi}\frac{\partial \gamma}{\partial p}\mathrm{d}p\mathrm{d}\Omega. \tag{20}$$

Comparing to the linearized system of incompressible Stokes equations, the additional terms related with plastic dilation are (terms with β and/or $\bar{\xi}$):

$$\text{bottom left:} \qquad \int_\Omega q\bar{\xi}\frac{\partial \gamma}{\partial \tilde{\boldsymbol{\varepsilon}}} : \mathrm{d}\tilde{\boldsymbol{\varepsilon}}\mathrm{d}\Omega, \tag{21}$$

$$\text{bottom right:} \qquad -\int_\Omega \frac{q\beta}{\Delta t}\mathrm{d}\Omega + \int_\Omega q\bar{\xi}\frac{\partial \gamma}{\partial p}\mathrm{d}p\mathrm{d}\Omega. \tag{22}$$

4 The Effective Viscosity

To calculate the additional terms, we need to know the derivatives of γ with respect to $\tilde{\boldsymbol{\varepsilon}}$ and p. As for the differentiation of viscosity, we use a finite difference approximation to calculate $\mathrm{d}\gamma$:

$$\begin{aligned} \left.\frac{\partial \gamma}{\partial \tilde{\boldsymbol{\varepsilon}}}\right|_{(\tilde{\boldsymbol{\varepsilon}},p)} &\approx \frac{\gamma(\tilde{\boldsymbol{\varepsilon}} + \delta\tilde{\boldsymbol{\varepsilon}}, p) - \gamma(\tilde{\boldsymbol{\varepsilon}}, p)}{\delta\tilde{\boldsymbol{\varepsilon}}}, \\ \left.\frac{\partial \gamma}{\partial p}\right|_{(\tilde{\boldsymbol{\varepsilon}},p)} &\approx \frac{\gamma(\tilde{\boldsymbol{\varepsilon}}, p + \delta p) - \gamma(\tilde{\boldsymbol{\varepsilon}}, p)}{\delta p}. \end{aligned} \tag{23}$$

The expression of γ in terms of $\tilde{\boldsymbol{\varepsilon}}$ and p can be derived as follows: in computation, we first assume that plastic yielding occur does not occur and then calculate a trial stress state $\boldsymbol{\sigma}^{\mathrm{tr}} = \boldsymbol{\tau}^{\mathrm{tr}} - p^{\mathrm{tr}}\boldsymbol{I}$ accordingly. If the value of $\Phi^{\mathrm{tr}} := \tau_{\mathrm{II}}^{\mathrm{tr}} - \xi p^{\mathrm{tr}} - \zeta$ is greater than zero, we map the stress state onto the yielding envelop according to the plastic flow rule. Substitution of Eqs. (10) and (12) in Eq. (8) gives

$$\begin{aligned} \Phi =& \tau_{\mathrm{II}} - \xi p - \zeta \\ =& 2\eta^{\mathrm{ve}}\tilde{\varepsilon}_{\mathrm{II}}\left(1 - \frac{\gamma}{2\tilde{\varepsilon}_{\mathrm{II}}}\right) - \xi\left(p^{\mathrm{tr}} + \frac{\gamma\bar{\xi}\Delta t}{\beta}\right) - \zeta \\ =& \Phi^{\mathrm{tr}} - \gamma\left(\eta^{\mathrm{ve}} + \frac{\xi\bar{\xi}\Delta t}{\beta}\right) \\ =& 0, \end{aligned} \tag{24}$$

from which we get

$$\gamma = \frac{\Phi^{\mathrm{tr}}}{\eta^{\mathrm{ve}} + \xi\bar{\xi}\Delta t/\beta}. \tag{25}$$

In practice, we often define an "effective" viscosity η^{eff} as

$$\eta^{\mathrm{eff}} := \frac{\tau_{\mathrm{II}}}{2\tilde{\varepsilon}_{\mathrm{II}}} = \eta^{\mathrm{ve}}\left(1 - \frac{\gamma}{2\tilde{\varepsilon}_{\mathrm{II}}}\right) = \eta^{\mathrm{ve}}\left[1 - \frac{\eta^{\mathrm{ve}} - (\xi p^{\mathrm{tr}} + \zeta)/2\tilde{\varepsilon}_{\mathrm{II}}}{\eta^{\mathrm{ve}} + \xi\bar{\xi}\Delta t/\beta}\right] = \frac{\xi\bar{\xi}\Delta t/\beta + (\xi p^{\mathrm{tr}} + \zeta)/2\tilde{\varepsilon}_{\mathrm{II}}}{1 + \xi\bar{\xi}\Delta t/(\beta\eta^{\mathrm{ve}})}. \tag{26}$$

When $\bar{\xi} = 0$, we have $\eta^{\mathrm{eff}} = (\xi p^{\mathrm{tr}} + \zeta)/2\tilde{\varepsilon}_{\mathrm{II}}$, which is the common expression of the effective viscosity.

Remark 1. *If we impose plastic dilation on an incompressible material, i.e.* $\beta = 0$ *and* $\bar{\xi} > 0$*, then the effective viscosity becomes*

$$\eta^{\mathrm{eff}} \approx \frac{\xi\bar{\xi}\Delta t/\beta}{\xi\bar{\xi}\Delta t/(\beta\eta^{\mathrm{ve}})} = \eta^{\mathrm{ve}},$$

which implies that no plastic yielding can take place under this situation.