## 1. Introduction

The problem of localization in time and frequency has always been of serious concern in modern physics because one of the major issues in applications is to analyze signals on different time-frequency domains and therefore to concentrate and localize signals on these domains. To be able to represent the frequency behavior of a signal locally in time, one has to consider the so called time-frequency localization operators. A variety of methods have been invented to construct such class of operators [1]. Coherent states (CS) are the natural tool in constructing phase space localization operators and have been extensively encountered in theoretical physics, in quantum mechanics and in many different areas of mathematical physics. Precisely, CS provide a close connection between classical and quantum formalisms so as to play a central role in the semi classical analysis. In general, they may be defined as an overcomplete family of normalized ket vectors $|\zeta\rangle$ which are labeled by points $\zeta$ of a phase-space domain $X$, belonging to a Hilbert space $\mathcal{H}$ that corresponds to a specific quantum model and provide $\mathcal{H}$ with a resolution of its identity operator as

$$\mathbf{1}_{\mathcal{H}} = \int_X |\zeta\rangle\langle\zeta| d\mu(\zeta). \tag{1.1}$$

with respect to a suitable integration measure $d\mu(\zeta)$ on $X$. These states are constructed in different ways. For an overview of all aspects of the theory of coherent states and their genesis, we refer to the [2, 4].

Equation (1.1) allows to implement a CS frame quantization [3] of the set of parameters $\zeta \in X$ by associating to a complex-valued function $\zeta \mapsto F(\zeta)$, satisfying appropriate conditions, the following operator on $\mathcal{H}$ :

$$F(\zeta) \mapsto P_F := \int\limits_X |\zeta\rangle\langle\zeta| F(\zeta) d\mu(\zeta). \tag{1.2}$$

If $F(\zeta)$ is semi-bounded real-valued function, the Friedrich extension [5] allows us to define $P_F$ as a self-adjoint operator. In particular, when $F = \chi_\Omega$ is the indicator function for some domain $\Omega$ in the phase space $X$, the resulting operator $P_{\chi_\Omega}$ is called a localization operator.

By using the CS of the harmonic oscillator, Daubechies [6] has discussed the localization operator $P_{\chi_\Omega}$ with $\Omega \subset \mathbb{C}$ being a disk of radius $\rho > 0$ by giving its eigenfunctions in terms of Hermite polynomials, and by expressing its discrete eigenvalues $\{\lambda_k^\rho\}$ in term of incomplete Gamma functions. She also has established the asymptotic behavior of these eigenvalues for varying $k = 0, 1, 2, ...$, and $\rho > 0$, and has given an estimate for the phase-content outside the localization domain $\Omega$.

In this paper, we deal with similar questions for the pseudo-harmonic oscillator

$$H_B = \frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2 + \frac{(2B-1)^2 - \frac{1}{4}}{x^2}\right] + (1 - B), \qquad 2B > 1 \tag{1.3}$$

acting on the Hilbert space $L^2(\mathbb{R}_+)$, whose importance consists in the fact that it is a solvable model and being, in a certain sense, an intermediate potential between the three dimensional harmonic oscillator potential and other anharmonic potentials such as Poschl-Teller or Morse potential [7, 8]. The $L^2$ eigenfunctions (number states) of $H_B$, which here are denoted by the ket vectors $\left|\ell_j^B\right\rangle$, may be superposed to perform a set of coherent states within the so-called *Hilbertian probabilistic scheme* (see [3]for the general theory) by choosing a set of analytic coefficients $C_j^B(z)$ on the complex unit disk $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ such that the associated photon-counting statistics follows a negative probability distribution. Such a CS are known as the negative binomials states (NBS) [9]. One interest on them is that they intermediate between pure coherent states and pure thermal states [10]and reduce to Susskind-Glogower phases states for a particular limit of the parameter [11] Beside, such coefficients $C_j^B(z)$, turn out to be basis elements of the weighted Bergman space, here denoted $\mathcal{A}^B(\mathbb{D})$, of analytic functions $g$ on $\mathbb{D}$, satisfying the growth condition $\int_{\mathbb{D}} |g(z)|^2(1-\bar{z}z)^{2B-2}d\eta(z) < +\infty$, where $d\eta$ denotes the Lebesgue measure on $\mathbb{D}$.

Our aim is, firstly, to show that these NBS which are labeled by points of the disk $\mathbb{D}$ can be retreived from the affine CS via the Cayley transform. We also link them to the Landau problem in the Poincaré upper half-plane. This connection may be exploited to generalize the obtained results to higher hyperbolic Landau levels. Secondly, we proceed by a quantization method based on these NBS in order to construct a phase space localization operator $P_R$ corresponding to the disk $D_R = \{z \in \mathbb{C}, |z| < R\}$ with $R < 1$, which stands for the quantum counterpart of the classical observable defined as the indicator function of the disk $D_R$. Precisely, we discuss some spectral properties of the operator $P_R$ such as its eigenvalues and their associated eigenfunctions in $L^2(\mathbb{R}_+)$. The expression of these eigenvalues together with the discrete spectral resolution of $P_R$ amount to a formula expressing this operator as a function of the Hamiltonian operator $H_B$ in (1.3). We also give an estimate for the phase space content of $P_R$ outside the domain $D_R$ in terms of the photon-counting probability distribution associated with the NBS. Moreover, this operator may be unitarily intertwined as $W \circ P_R \circ W^{-1} = \widetilde{P}_R$ via the second Bargmann transform $W$ associated with the NBS. This allows us to obtain the integral kernel of $\widetilde{P}_R$ when acting on the space $\mathcal{A}^B(\mathbb{D})$ by using calculations based on properties of some different hypergeometric functions.

The paper is organized as follows. In section 2, we recall the affine coherent states from which we derive the NBS.The connection with the Landau problem on the Poincaré upper half-plane is also pointed out. Section 3 deals with the coherent states quantization method. In particular, eigenvalues of the quantum counterpart with radial classical observables are obtained. For the indicator function of the disk $D_R$ we also provide these eigenvalues with a probabilistic interpretation and we discuss their extensions to hyperbolic higher Landau levels. In section 4, we give an estimate of the phase space content of the localization operator outside the disk $D_R$ in terms of the photon-counting probablity distribution. Section 5, we deal with the transfer of the localization operator to the weigthed Bergman space $\mathcal{A}^B(\mathbb{D})$ and to the calculation of its integral kernel.

## 2. Negative binomial states and the $B$-weight Maass Laplacian

2.1. **Affine coherent states.** We recall that the affine group is the set $\mathbf{G} = \mathbb{R} \times \mathbb{R}^{+}$, endowed with group law $(x, y) \cdot (x', y') = (x + yx', yy')$. $\mathbf{G}$ is a locally compact group with the left Haar measure $d\nu(x, y) = y^{-2} dx dy$. We shall consider one of the two inequivalent infinite dimensional irreducible unitary representations of the affine group $\mathbf{G}$, denoted $\pi_{+}$, realized on the Hilbert space $\mathcal{H} := L^{2}(\mathbb{R}^{+}, \xi^{-1} d\xi)$ as

$$\pi_{+}(x, y)[\varphi](\xi) := e^{\frac{1}{2} i x \xi} \varphi(y\xi), \qquad \varphi \in \mathcal{H}, \quad \xi > 0. \tag{2.1}$$

This representation is square integrable since it is easy to find a vector $\phi_0 \in \mathcal{H}$ such that the function $(x, y) \mapsto \langle \pi_{+}(x, y)[\phi_0], \phi_0 \rangle_{\mathcal{H}}$ belongs to $L^{2}(\mathbf{G}, d\nu)$. This condition can also be expressed by saying that the self-adjoint operator $\delta : \mathcal{H} \to \mathcal{H}$ defined as $\delta[\varphi](\xi) = \xi^{-\frac{1}{2}} \varphi(\xi)$ gives

$$\int_{\mathbf{G}} \langle \varphi_1, \pi_{+}(x, y)[\psi_1] \rangle \langle \pi_{+}(x, y)[\varphi_2], \psi_2 \rangle \, d\nu(x, y) = \langle \varphi_1, \varphi_2 \rangle \left\langle \delta^{\frac{1}{2}}[\varphi_1], \delta^{\frac{1}{2}}[\varphi_2] \right\rangle \tag{2.2}$$

for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in \mathcal{H}$. The operator $\delta$ is unbounded because $\mathbf{G}$ is not unimodular [12].

Keeping the condition $2B > 1$, we consider a set of CS labeled by elements $(x, y) \in \mathbf{G}$, which are obtained by acting, via the representation operator $\pi_{+}(x, y)$, on the admissible vector

$$\phi_B(\xi) := \frac{1}{\sqrt{2B}} \xi^{B} e^{-\frac{1}{2}\xi}, \quad \xi > 0. \tag{2.3}$$

Precisely,

$$\left| \tau_{(x,y),B} \right\rangle := \pi_{+}(x, y)[\phi_B] \tag{2.4}$$

and satisfy the resolution of the identity operator

$$\mathbf{1}_{\mathcal{H}} = c_B \int_{\mathbf{G}} d\mu(x, y) \left| \tau_{(x,y),B} \right\rangle \left\langle \tau_{(x,y),B} \right| \tag{2.5}$$

where $c_B := 2B - 1$ and the Dirac's bra-ket notation $|\Phi\rangle\langle\Phi|$ means the rank-one operator $\phi \longmapsto \langle \Phi, \phi \rangle_{\mathcal{H}} . \Phi$ with $\Phi, \phi \in \mathcal{H}$. In the $\xi$-coordinate, wavefunctions of CS defined by Eq. (2.4) read

$$\langle \xi | \, \tau_{(x,y),B} \rangle = \frac{1}{\sqrt{2B}} (\xi y)^{B} e^{-\frac{1}{2}\xi(y - ix)}, \qquad \xi > 0, \tag{2.6}$$

and are known as the affine CS [13].

2.2. **Connection with the $B$-weight Maass Laplacian.** To describe the connection of CS (2.6) with the lowest hyperbolic Landau level, we may first identify the affine group $\mathbf{G}$ with the Poincaré upper half-plane $\mathbb{H}^{2} = \{x + iy, x \in \mathbb{R}, y > 0\}$. Then, to these CS we may attach, as usual, the CS transform $\mathcal{B}_0 : \mathcal{H} \to L^{2}(\mathbb{H}^{2}, d\nu)$ defined by [14]:

$$\mathcal{B}_0[\phi](x, y) = \sqrt{c_B} \int_{0}^{+\infty} \overline{\langle \xi | \, \tau_{(x,y),B} \rangle} \phi(\xi) \xi^{-1} d\xi \tag{2.7}$$

whose range is the eigenspace of the $B$-weight Maass Laplacian

$$\Delta_B = y^2\left(\partial_x^2 + \partial_y^2\right) - 2iBy\partial_x, \tag{2.8}$$

associated with the eigenvalue

$$\epsilon_m^B = (B-m)(1-B+m), \quad m = 0, 1, ..., \left\lfloor B - \frac{1}{2} \right\rfloor, \tag{2.9}$$

where $\lfloor a \rfloor$ denotes the greatest integer not exceeding $a$. We precisely have

$$\mathcal{B}_0[\mathcal{H}] \equiv \left\{ f \in L^2\left(\mathbb{H}^2, d\nu\right),\ \Delta_B f = \epsilon_0^B f \right\}. \tag{2.10}$$

The operator $\Delta_B$ also stands (in suitable units and up to an additive constant) for the Schrödinger operator describing the dynamics of a charged particle moving on $\mathbb{H}^2$ under the action of a magnetic field of strength proportional to $B$. This is an elliptic densely defined operator on the Hilbert space $L^2(\mathbb{H}^2, d\nu)$, with a unique self-adjoint realization also denoted by $\Delta_B$. Its spectrum consists of two parts: a continuous part $\left[\frac{1}{4}\theta, +\infty\right[$, corresponding to scattering states and the finite number of eigenvalues $\epsilon_m^B$ each one with infinite degeneracy, called hyperbolic Landau levels . Finally, the reproducing kernel of the Hilbert space $\mathcal{B}_0[\mathcal{H}]$ can be obtained from the overlapping function $\langle \tau_{w,B}, \tau_{\zeta,B}\rangle_{\mathcal{H}}$ between two CS as

$$K_0^B(w,\zeta) = \left(\frac{\left|w-\bar{\zeta}\right|^2}{4\mathrm{Im}w\mathrm{Im}\zeta}\right)^{-B}\left(\frac{\zeta-\bar{w}}{w-\bar{\zeta}}\right)^B, \quad w,\zeta \in \mathbb{H}^2. \tag{2.11}$$

2.3. **Negative binomial states.** We can write a version of these CS as states labeled by points $z$ of the unit disk $\mathbb{D}$ by using the inverse Cayley transform $\mathcal{C}^{-1} : \mathbb{D} \to \mathbf{G}$ given by

$$\mathcal{C}^{-1}(z) = \left(-2\frac{\mathrm{Im}z}{|1-z|^2}, \frac{1-|z|^2}{|1-z|^2}\right), \quad z \in \mathbb{D}. \tag{2.12}$$

Indeed, we may still define this version as states in $\mathcal{H}$ as

$$\kappa_{z,B} := \left(\frac{1-\bar{z}}{1-z}\right)^B \pi_+\left(\mathcal{C}^{-1}(z)\right)[\phi_B]. \tag{2.13}$$

Direct calculations lead to their wave functions in the $\xi$-coordinate as

$$\langle \xi | \kappa_{z,B}\rangle = \frac{1}{\sqrt{\Gamma(2B)}}\left(\frac{(1-z\bar{z})}{(1-z)^2}\xi\right)^B \exp\left(-\frac{1}{2}\left(\frac{1+z}{1-z}\right)\xi\right), \qquad \xi > 0. \tag{2.14}$$

The latter ones may slightly be modified in order to perform them as vectors of $L^2(\mathbb{R}_+, d\xi)$ labeled by points of the disk $\mathbb{D}$ as

$$\langle \xi | \widetilde{\kappa}_{z,B}\rangle := \sqrt{\frac{2}{\xi}}\langle \xi^2 | \kappa_{z,B}\rangle \tag{2.15}$$

These states obey the normalization condition $\langle \widetilde{\kappa}_{z,B}, \widetilde{\kappa}_{z,B}\rangle_{L^2(\mathbb{R}_+)} = 1$ and satisfy the resolution of the identity operator as

$$1_{L^2(\mathbb{R}_+)} = \int_{\mathbb{D}} |\widetilde{\kappa}_{z,B}\rangle\langle \widetilde{\kappa}_{z,B}|\, d\eta_B(z), \tag{2.16}$$

with respect to the measure

$$d\eta_B(z) := \frac{(2B-1)}{\pi\left(1-z\overline{z}\right)^2} d\eta(z). \tag{2.17}$$

Now, as for the canonical CS of the harmonic oscillator, we seek for a number states expansion of the CS $|\widetilde{\kappa}_{z,B}\rangle$ in terms of the analytic coefficients

$$C_j^B(z) := (2B-1)^{1/2}\sqrt{\frac{\Gamma(2B+j)}{j!\Gamma(2B)}} z^j, \quad j = 0, 1, 2, ..., \tag{2.18}$$

which constitute an orthonormal basis of the weighted Bergman space $\mathcal{A}^B(\mathbb{D})$. For that, we start from the expression of the CS, $|\widetilde{\kappa}_{z,B}\rangle$ as given by (2.15) – (2.14) and we make use of the generating function for the Laguerre polynomials ([15], p.239):

$$\sum_{j=0}^{+\infty} t^j L_j^{(\alpha)}(x) = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-t}{1-t}x\right), \quad \alpha > -1. \tag{2.19}$$

We obtain, after some calculations, the series expansion

$$\langle \xi \,|\widetilde{\kappa}_{z,B}\rangle = \left(\frac{2B-1}{(1-z\bar{z})^{2B}}\right)^{-1/2} \sum_{j=0}^{+\infty} C_j^B(z)\ell_j^B(\xi) \tag{2.20}$$

where $\ell_j^B(\xi)$ are the Laguerre functions

$$\ell_j^B(\xi) := \left(\frac{2j!}{\Gamma(2B+j)}\right)^{1/2} \xi^{2B-\frac{1}{2}} e^{-\frac{1}{2}\xi^2} L_j^{(2B-1)}(\xi^2), \quad j = 0, 1, 2, ..., \tag{2.21}$$

which are known to constitute a complete orthonormal system in $L^2(\mathbb{R}_+, d\xi)$.

**Remark**

The CS (2.14) also coincide with those constructed by Molanar *et al* [16] for the Morse potential [17] by an algebraic way based on supersymetry and shape invariance properties where the shape parameter may be taken as our $B > 0$. For this potential, they first were were introduced by Nieto *et al* [18] as generalized minimal uncertainty states.

## 3. Quantization via NBS $|\widetilde{\kappa}_{z,B}\rangle$

The resolution of the identity (2.16) allows to implement a CS or frame quantization [4] of the set of parameters $\mathbb{D}$ by associating to a function $\mathbb{D} \ni z \mapsto F(z,\overline{z}) \in \mathbb{C}$ that satisfies appropriate conditions, the following operator in $L^2(\mathbb{R}_+, d\xi)$ :

$$F \mapsto \wp_F^B := \int_{\mathbb{D}} |\widetilde{\kappa}_{z,B}\rangle \langle\widetilde{\kappa}_{z,B}|\, F(z,\overline{z}) \frac{(2B-1)}{\pi(1-z\bar{z})^2} d\eta(z). \tag{3.1}$$

The Friedrich extension [5] allows to define $\wp_F^B$ as a self-adjoint operator if $F$ is a semi-bounded real-valued function.

In order to proceed with the quantization through the CS $|\widetilde{\kappa}_{z,B}\rangle$ along the linear map (3.1), we may substitute the expression (2.20) into the integral form in (3.1). This gives the expression

$$\wp_F^B = \sum_{j,k=0}^{+\infty} \sqrt{\frac{\Gamma(2B+j)}{\pi j!\Gamma(2B)}} \sqrt{\frac{\Gamma(2B+k)}{\pi k!\Gamma(2B)}} \left[ (2B-1) \int_{\mathbb{D}} z^k \bar{z}^j (1-z\bar{z})^{2B-2} F(z,\bar{z}) d\eta(z) \right] \left|\ell_j^B\right\rangle \left\langle \ell_k^B \right| \tag{3.2}$$

which represents its disrete spectral resolution

$$\wp_F^B = \sum_{j,k=0}^{+\infty} \left[\gamma_F^B\right]_{j,k} |\ell_k^B\rangle\langle\ell_j^B| \tag{3.3}$$

where the matrix elements are (at least formally) given by

$$\left[\gamma_F^B\right]_{j,k} = \frac{1}{\pi\Gamma(2B-1)} \left(\frac{\Gamma(2B+j)\Gamma(2B+k)}{j!k!}\right)^{1/2} \int_{\mathbb{D}} \bar{z}^j z^k (1-z\bar{z})^{2B-2} F(z,\bar{z}) d\eta(z) \tag{3.4}$$

and $\left\{\ell_j^B\right\}$ is the orthonormal basis of $L^2(\mathbb{R}_+, d\xi)$, which is given in (2.21).

3.1. **Radial classical observables.** For a radial weight function $F$, the above discrete spectral resolution of $\wp_F^B$ leads to more precise expressions for its eigenvalues. Indeed, by setting $F(z,\bar{z}) = F(r^2)$, $r = |z|$ and using polar coordinates in the expression (3.4) of matrix elements, we get that

$$\left[\gamma_F^B\right]_{j,k} = \frac{1}{\pi\Gamma(2B-1)} \left(\frac{\Gamma(2B+j)\Gamma(2B+k)}{j!k!}\right)^{1/2} \int_0^{2\pi}\int_0^1 r^k r^j e^{ik\theta} e^{-ij\theta} (1-r^2)^{2B-2} F(r^2) r dr d\theta. \tag{3.5}$$

By the fact that

$$\int_0^{2\pi} e^{i(k-j)\theta} d\theta = 2\pi\delta_{k,j}, \quad j,k = 0,1,2,..., \tag{3.6}$$

on can easily see that only the case $j = k$ produces a non zero matrix element as

$$\left[\gamma_F^B\right]_{j,j} = \frac{\Gamma(2B+j)}{\Gamma(2B-1)\Gamma(j+1)} \int_0^1 \rho^j (1-\rho)^{2B-2} F(\rho) d\rho. \tag{3.7}$$

By writing the prefactor as $(\mathcal{B}(j+1, 2B-1))^{-1}$, we obtain the expression of the $\lambda_j^{B,F}$ as

$$\lambda_j^{B,F} = \frac{1}{\mathcal{B}(j+1, 2B-1)} \int_0^1 \rho^j (1-\rho)^{2B-2} F(\rho) d\rho, \tag{3.8}$$

where $\mathcal{B}(a,b)$ denotes the Beta function with $a, b > 0$. The operator $\wp_F^B$ has the following discrete spectral resolution with respect to the orthonormal basis $\{\ell_j^B\}$

as

$$\wp_F^B = \sum_{j=0}^{+\infty} \lambda_j^{B,F} \left|\ell_j^B\right\rangle \left\langle \ell_j^B\right|, \tag{3.9}$$

and it's not difficult to check that

$$\wp_F^B \left[\ell_j^B\right] = \lambda_j^{B,F} \ell_j^B. \tag{3.10}$$

Note that (3.10) means that the operators $\wp_F^B$ and $H_B$ have $\{\ell_j^B\}$ as a commun set of eigenfunctions.

In particular, we here consider the disk $D_R := \{z \in \mathbb{C},\ |z| < R\}$ with $0 < R < 1$ and we choose as classical observable the indicator function of this domain. By putting $F(r^2) = 1$ if $r < R$ and $F(r^2) = 0$ if $r \geq R$, the formula (3.8) takes the form

$$\lambda_j^{B,R} = \frac{1}{\mathcal{B}(j+1, 2B-1)} \int_0^{R^2} \rho^j (1-\rho)^{2B-2} d\rho = \mathcal{I}_{R^2}(j+1, 2B-1) \tag{3.11}$$

where

$$\mathcal{I}_x(a,b) = \frac{1}{\mathcal{B}(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad 0 < x < 1, \tag{3.12}$$

is the regularized incomplete Beta function. In view of (3.9), the discrete spectral resolution of $\wp_F^B$ reads

$$\wp_R^B = \sum_{j=0}^{+\infty} \mathcal{I}_{R^2}(j+1, 2B-1) \left|\ell_j^B\right\rangle \left\langle \ell_j^B\right|. \tag{3.13}$$

By another side, the vector basis $\ell_j^B$ are eigenfunctions of $H_B$ while acting on $L^2(\mathbb{R}_+, d\xi)$. Precisely,

$$H_B\left[\ell_j^B\right] = (j+1)\ell_j^B,\ j = 0, 1, 2, \dots\ . \tag{3.14}$$

Therefore, we may write $\wp_R^B$ as a function of $H_B$ as

$$\wp_R^B = \mathcal{I}_{R^2}(j+1, 2B-1). \tag{3.15}$$

3.2. **A probabilistic representation for eigenvalues $\lambda_j^{B,R}$.** We note that the eigenvalues (3.11) may also be written as

$$\lambda_j^{B,R} = \| C_j^B 1_{\mathbb{D}_R} \|^2_{L^2(\mathbb{D},(1-z\bar{z})^{2B-2}d\eta)} = \int_0^{R^2} \mathfrak{g}_{j,B}(\rho) d\rho \tag{3.16}$$

where

$$\mathfrak{g}_{j,B}(\rho) = \frac{\Gamma(2B+j)}{\Gamma(2B-1)\Gamma(j+1)} (1-\rho)^{2B-2} \rho^j, \quad 0 \leq \rho < 1 \tag{3.17}$$

which turns out to be the identity function of the Beta distribution $\mathcal{Y}_{j,B}^{(0)} \sim \mathcal{B}e(j+1, 2B-1)$ whose characteristic function is known to be given by the confluent hypergeometric series as $u \mapsto {}_1F_1(j+1, 2B+1; iu)$ with $i^2 = -1$.Therefore,

$$\lambda_j^{B,R} = \Pr(\mathcal{Y}_{j,B}^{(0)} \leq R^2) \tag{3.18}$$

would provide us with a probabilistic representation of these eigenvalues.

For higher hyperbolic Landau levels, the generalized form of the density function (3.16) is given by

$$\mathfrak{g}_{B,j}^{(m)}(\rho) := (2B-2m-1)\,\frac{(m\wedge j)!}{(m\vee j)!}\frac{\Gamma(2B-2m+m\vee j)}{\Gamma(2B-2m+m\wedge j)}(1-\rho)^{2B-2m-2}\rho^{|m-j|} \times \left(P_{m\wedge j}^{(|m-j|,2(B-m)-1)}\left(1-2\rho\right)\right)^2 \tag{3.19}$$

where $P_k^{(\alpha,\beta)}\,(.)$ is a Jacobi polynomial [15] and $m=0,1,...,\left\lfloor B-\frac{1}{2}\right\rfloor$. Let us denote by $\mathcal{Y}_{j,B}^{(m)}$ the random variable having $\rho\mapsto\mathfrak{g}_{j,B}^{(m)}(\rho)$ as its density, then

$$\lambda_j^{B,R,m} := \Pr\left(\mathcal{Y}_{j,B}^{(m)}\leq R^2\right) = \int_0^{R^2}\mathfrak{g}_{j,B}^{(m)}(\rho)d\rho \tag{3.20}$$

would provide us with the probabilistic representation of eigenvalues $\lambda_j^{B,R,m}$ of the restricted operator $\mathfrak{K}_{B,m}\mid_{D_R}$to the disk $D_R$, where $\mathfrak{K}_{B,m}$ is the projection operator onto the eigenspace

$$\mathcal{E}_{B,m}\left(\mathbb{D}\right) = \left\{f\in L^2\left(\mathbb{D},\left(1-z\overline{z}\right)^{2B-2}d\eta\left(z\right)\right),\widetilde{\Delta}_Bf=\sigma_{B,m}f\right\} \tag{3.21}$$

of the $B$-weight Maass Laplacian

$$\widetilde{\Delta}_B = -4\left(1-z\overline{z}\right)\left(\left(1-z\overline{z}\right)\frac{\partial^2}{\partial z\partial\overline{z}}-2B\overline{z}\frac{\partial}{\partial\overline{z}}\right), \tag{3.22}$$

associated with the hyperbolic Landau level

$$\sigma_{B,m} = 4m\left(2B-m-1\right),\quad m=0,1,...,\left\lfloor B-\frac{1}{2}\right\rfloor. \tag{3.23}$$

So that the integral kernel of operator $\mathfrak{K}_{B,m}$ is the reproducing kernel

$$K_{B,m}\left(z,w\right) = \pi\left(2B-2m-1\right)\left(1-z\overline{w}\right)^{-2B}\left(\frac{\left|1-z\overline{w}\right|^2}{\left(1-z\overline{z}\right)\left(1-w\overline{w}\right)}\right)^m \times P_m^{(0,2(B-m)-1)}\left(2\frac{\left(1-z\overline{z}\right)\left(1-w\overline{w}\right)}{\left|1-z\overline{w}\right|^2}-1\right) \tag{3.24}$$

of the eigenspace (3.24). Note that for $2B>1$ and $m=0$ this eigenspace reduces to the weighted Bergman space of analytic functions $g$ on $\mathbb{D}$, satisfying the growth condition $\int_{\mathbb{D}}|g(z)|^2(1-\bar{z}z)^{2B-2}d\eta(z)<+\infty$. In other words, $\mathcal{E}_{B,0}\left(\mathbb{D}\right)\equiv\mathcal{A}^B\left(\mathbb{D}\right)$. This remark makes possible to extend our analysis to the above higher hyperbolic Landau levels $\sigma_{B,m}$. In this respect the connection with the results in [19] may be useful.

## 4. Phase space content of $\wp_R^B$ outside $D_R$

Note that the phase space cutoff by the operator $\wp_R^B$ is not sharp in the sense that it will have some phase space content outside the localization domain $D_R$. This is illustrated by the fact that at least for some coherence point $z_0 \in \mathbb{D} \setminus D_R$ we have

$$\langle \widetilde{\kappa}_{z_0,B} \left| \wp_R^B[f] \right\rangle_{L^2(\mathbb{R}_+)} \neq 0, \quad f \in L^2(\mathbb{R}_+). \tag{4.1}$$

Precisely, we have the following estimate involving the photon counting statistics which obey the negative binomial probability distribution $\mathcal{X} \sim \mathcal{NB}(2B, z_0\bar{z}_0)$ with parameters $2B$ and $z_0\bar{z}_0$

$$\left| \langle \widetilde{\kappa}_{z_0,B} \left| \wp_R^B[f] \right\rangle_{L^2(\mathbb{R}_+)} \right| \leq \sqrt{\mathbb{E}\left( \left( \mathcal{I}_{R^2}\left( \mathcal{X}+1, 2B-1 \right) \right)^2 \right)} \|f\|_{L^2(\mathbb{R}_+)}. \tag{4.2}$$

To prove (4.2) we start by replacing in the scalar product (4.1) the operator $\wp_R^B$ by its discrete spectral resolution (3.13) as

$$\left\langle \widetilde{\kappa}_{z_0,B}, \left( \sum_{j=0}^{+\infty} \lambda_j^{B,R} \left| \ell_j^B \right\rangle \left\langle \ell_j^B \right| \right) [f] \right\rangle_{L^2(\mathbb{R}_+)} = \sum_{j=0}^{+\infty} \lambda_j^{B,R} \left\langle \widetilde{\kappa}_{z_0,B} | \ell_j^B \right\rangle_{L^2(\mathbb{R}_+)} \left\langle \ell_j^B | f \right\rangle_{L^2(\mathbb{R}_+)}. \tag{4.3}$$

Recalling the number states expansion (2.20) of $|\widetilde{\kappa}_{z,B}\rangle$, we may write

$$\left\langle \widetilde{\kappa}_{z_0,B} | \ell_j^B \right\rangle_{L^2(\mathbb{R}_+)} = (1 - z_0\bar{z}_0)^B \sqrt{\frac{\Gamma(2B+j)}{j!\Gamma(2B)}} z_0^j. \tag{4.4}$$

Therefore (4.3) reads

$$\langle \widetilde{\kappa}_{z_0,B} \left| \wp_R^B[f] \right\rangle_{L^2(\mathbb{R}_+)} = (1 - z_0\bar{z}_0)^B \sum_{j=0}^{+\infty} \lambda_j^{B,R} \left( \frac{\Gamma(2B+j)}{j!\Gamma(2B)} \right)^{1/2} \left\langle \ell_j^B | f \right\rangle_{L^2(\mathbb{R}_+)} z_0^j. \tag{4.5}$$

$$= (1 - z_0\bar{z}_0)^B \left\langle \sum_{j=0}^{+\infty} \lambda_j^{B,R} \left( \frac{\Gamma(2B+j)}{j!\Gamma(2B)} \right)^{1/2} z_0^j \ell_j^B | f \right\rangle_{L^2(\mathbb{R}_+)}. \tag{4.6}$$

Setting

$$\vartheta := \sum_{j=0}^{+\infty} \lambda_j^{B,R} \left( \frac{\Gamma(2B+j)}{j!\Gamma(2B)} \right)^{1/2} z_0^j \ell_j^B, \tag{4.7}$$

and using the Cauchy-Bunyakovsky-Schwarz inequality

$$\left| \langle \widetilde{\kappa}_{z_0,B} | \wp_R^B[f] \rangle_{L^2(\mathbb{R}_+)} \right| \leq |1 - z_0\bar{z}_0|^B \|\vartheta\|_{L^2(\mathbb{R}_+)} \|f\|_{L^2(\mathbb{R}_+)}. \tag{4.8}$$

Now, the $L^2$ norm of $\vartheta$ may be written

$$|1 - z_0\bar{z}_0|^B \|\vartheta\|_{L^2(\mathbb{R}_+)} = \left( \sum_{j=0}^{+\infty} \left( \lambda_j^{B,R} \right)^2 \left[ \frac{\Gamma(2B+j)}{j!\Gamma(2B)} (z_0\bar{z}_0)^j (1 - z_0\bar{z}_0)^{2B} \right] \right)^{1/2}. \tag{4.9}$$

By recognizing in (4.8) the probability distribution of $\mathcal{X} \sim \mathcal{NB}(2B, z_0\overline{z_0})$ :

$$\Pr\left( \mathcal{X} = j \right) = \frac{\Gamma(2B+j)}{j!\Gamma(2B)} \left( z_0\overline{z_0} \right)^j \left( 1 - z_0\overline{z_0} \right)^{2B}, \quad j = 0, 1, 2, \cdots, \tag{4.10}$$

and using the expression of the eigenvalues $\lambda_j^{B,R}$, Eq.(4.8) takes the form

$$|1-z_0\bar{z}_0|^B\|\vartheta\|_{L^2(\mathbb{R}_+)}=\left(\sum_{j=0}^{+\infty}\left(\mathcal{I}_{R^2}(j+1,2B-1)\right)^2\Pr\left(\mathcal{X}=j\right)\right)^{1/2}. \tag{4.11}$$

Finally, the right hand side of (4.11) can be viewed as the square root of an expectation value as $\mathbb{E}\left(\left(\mathcal{I}_{R^2}(\mathcal{X}+1,2B-1)\right)^2\right)$.

## 5. Integral kernel of $\widetilde{\wp}_R^B$ on the Bergman space $\mathcal{A}^B(\mathbb{D})$

The CS transform associated with $|\widetilde{\kappa}_{z,B}\rangle$ is the isometric isomorphism $W_B: L^2(\mathbb{R}_+)\to\mathcal{A}^B(\mathbb{D})$ defined by $\varphi\mapsto W_B[\varphi](z):=(1-z\bar{z})^{-B}\langle\varphi\,|\widetilde{\kappa}_{z,B}\rangle_{L^2(\mathbb{R}_+)}$, which may also be called the second Bargmann transform [20]. Explicitly,

$$W_B[f](z)=\sqrt{\frac{2}{\Gamma(2B)}}(1-z)^{-2B}\int_0^{+\infty}\xi^{2B-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\frac{1+z}{1-z}\right)\xi^2\right)d\xi,\ \ z\in\mathbb{D}. \tag{5.1}$$

Now, to transfer the operator $\wp_R^B$ to be acting on functions $f\in\mathcal{A}^B(\mathbb{D})$, we use the relation $W_B\circ\wp_R^B\circ W_B^{-1}\equiv\widetilde{\wp}_R^B$. We, precisely, obtain that

$$\widetilde{\wp}_R^B[f](w)=\int_{\mathbb{D}}P_R^B(z,w)f(z)(1-z\bar{z})^{2B-2}d\eta(z) \tag{5.2}$$

where the integral kernel is given by

$$P_R^B(z,w)=\frac{(2B-1)^2R}{(1-R\bar{z}w)^{2B}}F_1\left(2-2B,1-2B,2B,2,R,\frac{R-R\bar{z}w}{1-R\bar{z}w}\right) \tag{5.3}$$

In particular, at the limit $R\to1$,

$$\lim_{R\to1}P_R^B(z,w)=\frac{(2B-1)}{(1-\bar{z}w)^{2B}}, \tag{5.4}$$

which is the reproducing kernel of the Bergman space $\mathcal{A}^B(\mathbb{D})$. Here, $F_1$ denotes the Appel hypergeometric double series

$$F_1\left(\alpha,\beta,\gamma;\omega;u,v\right)=\sum_{j,k=0}^{\infty}\frac{(\alpha)_{j+k}\,(\beta)_j\,(\gamma)_k}{j!k!\,(\omega)_{j+k}}u^jv^k,\,|u|<1,|v|<1.$$

To prove (5.2), let us take $f\in\mathcal{A}^B(\mathbb{D})$ and apply the inverse of $W_B$ as

$$W_B^{-1}[f](\xi)=\int_{\mathbb{D}}f(z)\,\langle\xi|\,\widetilde{\kappa}_{z,B}\rangle(1-z\bar{z})^{-B}d\eta_B(z),\ \ \xi>0. \tag{5.5}$$

Next, we proceed by the action of $\wp_R^B$ on $W_B^{-1}[f]$, which successively gives

$$\wp_R^B\left[W_B^{-1}[f]\right](y)=\sum_{k=0}^{+\infty}\lambda_k^{B,R}\left\langle\ell_k^B|\int_{\mathbb{D}}f(z)|\widetilde{\kappa}_{z,B}\rangle(1-z\bar{z})^{-B}d\eta_B(z)\right\rangle\langle y|\ell_k^B\rangle \tag{5.6}$$

$$= \sum_{k=0}^{+\infty} \lambda_k^{B,R} \left( \int_{\mathbb{R}_+} \overline{\ell_k^B(\xi)} \left( \int_{\mathbb{D}} f(z) \, \langle x | \, \widetilde{\kappa}_{z,B} \rangle (1 - z\bar{z})^{-B} d\eta_B(z) \right) d\xi \right) \ell_k^B(y) \tag{5.7}$$

$$= \sum_{k=0}^{+\infty} \lambda_k^{B,R} \int_{\mathbb{D}} f(z) \left( (1 - z\bar{z})^{-B} \overline{\int_{\mathbb{R}_+} \ell_k^B(\xi) \overline{\langle \xi | \, \widetilde{\kappa}_{z,B} \rangle} dx} \right) d\eta_B\,(z)\, \ell_k^B(y). \tag{5.8}$$

Note that the last equation may be rewritten as

$$\wp_R^B \left[ W_B^{-1}[f] \right](y) = \sum_{k=0}^{+\infty} \lambda_k^{B,R} \left[ \int_{\mathbb{D}} f(z) \overline{W_B[\ell_k^B](z)} d\eta_B(z) \right] \ell_k^B(y). \tag{5.9}$$

We again apply $W_B$ to (5.9) :

$$W_B \left[ \wp_R^B \left[ W_B^{-1}[f] \right] \right](w) = \sum_{k=0}^{+\infty} \lambda_k^{B,R} \left[ \int_{\mathbb{D}} f(z) \overline{W_B[\ell_k^B](z)} d\eta_B(z) \right] W_B[\ell_k^B](w). \tag{5.10}$$

Since $W_B[\ell_k^B](w) = C_k^B(w)$, then we obtain

$$\widetilde{\wp}_R^B[f](w) = \int_{\mathbb{D}} \left[ \sum_{k=0}^{+\infty} \lambda_k^{B,R} \overline{C_k^B(z)} C_k^B(w) \right] d\eta_B(z). \tag{5.11}$$

Hence, the kernel integral function is given by

$$P_R^B(z, w) = \sum_{k=0}^{+\infty} \lambda_k^{B,R} \overline{C_k^B(z)} C_k^B(w). \tag{5.12}$$

To write this kernel in a closed form, we recall (3.11) and (2.18), then (5.12) becomes

$$P_R^B(z, w) = (2B-1) \sum_{k=0}^{+\infty} \left[ \frac{1}{B(k+1, 2B-1)} \int_0^R \rho^k (1-\rho)^{2B-2} d\rho \right] \frac{\Gamma(2B+k)}{k!\Gamma(2B)} \bar{z}^k w^k \tag{5.13}$$

$$= (2B-1)^2 \sum_{k=0}^{+\infty} \frac{\Gamma(2B+k)\Gamma(2B+k)}{\Gamma(2B)\Gamma(2B)} \frac{1}{k!} \frac{(\bar{z}w)^k}{k!} \left( \int_0^R t^k (1-t)^{2B-2} dt \right) \tag{5.14}$$

$$= (2B-1)^2 \int_0^R (1-t)^{2B-2} \left( \sum_{k=0}^{+\infty} \frac{(2B)_k (2B)_k}{(1)_k} \frac{(t\bar{z}w)^k}{k!} \right) dt \tag{5.15}$$

The sum inside the integral can be presented as the Gauss hypergeometric function ${}_2F_1$ as

$$P_R^B(z, w) = (2B-1)^2 \int_0^R (1-t)^{2B-2} {}_2F_1(2B, 2B, 1, t\bar{z}w) dt. \tag{5.16}$$

By setting $2B = \alpha$, $\bar{z}w = \omega$ and

$$I_R = \int_0^R {}_2F_1(\alpha, \alpha, 1; \omega t)(1-t)^{\alpha-2} dt. \tag{5.17}$$

By making use of the formula [21, p.316] :

$$\int_0^y x^{c-1}(y-x)^{\beta-1}(1-xz)^{-\tau}{}_2F_1(a,b,c;wx)dx = \mathcal{B}(c,\beta)y^{c+\beta-1}(1-yz)^{-\tau}F_3(\tau,a,\beta,b,c+\beta,\frac{yz}{yz-1};wy), \tag{5.18}$$

$y, Re(c), Re(\beta) > 0; |arg(1-wy)|, |arg(1-z)| < \pi$, for parameters $y = R, x = t, c = 1, \beta = 1, z = 1, \tau = 2-2B, a = 2B, b = 2B$, the integral (5.17) takes the form

$$I_R = R(1-R)^{\alpha-2}F_3(2-\alpha,\alpha,1,\alpha,2;\frac{R}{R-1},\omega R) \tag{5.19}$$

Next, we may apply the transformation [21, p.450]:

$$F_3\left(a,a',b,b';a+a',w,z\right) = (1-z)^{-b'}F_1\left(a,b,b';a+a';w,\frac{z}{z-1}\right) \tag{5.20}$$

to rewrite the $F_3$ hypergeometric function in (5.19) as

$$F_3\left(2-\alpha,\alpha,1,\alpha,\frac{R}{R-1},\omega R\right) = (1-\omega R)^{-\alpha}F_1\left(2-\alpha,1,\alpha,2,\frac{R}{R-1},\frac{R\omega}{R\omega-1}\right). \tag{5.21}$$

Therefore, Eq.(5.19) reads

$$I_R = R(1-R)^{\alpha-2}(1-\omega R)^{-\alpha}F_1\left(2-\alpha,1,\alpha,2,\frac{R}{R-1},\frac{R\omega}{R\omega-1}\right). \tag{5.22}$$

By applying the transformation

$$F_1\left(a,b_1,b_2,c;X;Y\right) = (1-X)^{-b_1}(1-Y)^{-b_2}F_1\left(c-a,b_1,b_2;c,\frac{X}{X-1},\frac{Y}{Y-1}\right) \tag{5.23}$$

we may reduce $I_R$ as

$$I_R = R(1-R)^{\alpha-1}F_1\left(\alpha,1,\alpha,2;R,R\omega\right). \tag{5.24}$$

Next, by using the symmetry relation

$$F_1\left(a,b,b';c,z,u\right) = F_1\left(a,b',b;c,u,z\right) \tag{5.25}$$

together with the identity [21, p.449]

$$F_1\left(a,b,b';c,u,z\right) = (1-u)^{c-a-b}(1-z)^{-b'}F_1\left(c-a,c-b-b',b',c,u,\frac{u-z}{1-z}\right) \tag{5.26}$$

enable us to rewrite $I_R$ as

$$I_R = \frac{R}{(1-R\omega)^{2B}}F_1\left(2-2B,1-2B,2B,2,R,\frac{R-R\omega}{1-R\omega}\right) \tag{5.27}$$

Therefore, the kernel function (5.16) reads

$$P_R^B(z,w) = \frac{(2B-1)^2R}{(1-R\bar{z}w)^{2B}}F_1\left(2-2B,1-2B,2B,2,R,\frac{R-R\bar{z}w}{1-R\bar{z}w}\right). \tag{5.28}$$

To check the limit of this kernel as $R \to 1$, we first observe that

$$F_1\left(2-2B, 1-2B, 2B, 2; R, \frac{R-R\bar{z}w}{1-R\bar{z}w}\right) \to F_1\left(2-2B, 1-2B, 2B, 2, 1, 1\right) \text{ as } R \to 1. \tag{5.29}$$

By using the identity [21, p.452]

$$F_1\left(a, b, b', c; Z; Z\right) = {}_2F_1\left(a, b+b', c; Z\right), \tag{5.30}$$

the obtained expression limit in (5.29) reduces as

$$F_1\left(2-2B, 1-2B, 2B, 2B, 2; 1, 1\right) = {}_2F_1\left(2-2B, 1, 2; 1\right). \tag{5.31}$$

Finally, we use the Gauss theorem [21, p.489]:

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re(c-a-b)} > 0 \tag{5.32}$$

for parameters $a = 2-2B, b = 1$ and $c = 2$ to get that

$${}_2F_1\left(2-2B, 1, 2; 1\right) = \frac{1}{2B-1}. \tag{5.33}$$

This, leads to the limit

$$\lim_{R\to 1} P_R^B(z,w) = \frac{(2B-1)}{(1-\bar{z}w)^{2B}} \tag{5.34}$$

This completes the proof.

We end this section by observing that (5.8) may provides us with a family of Hilbert spaces (RKHS) indexed by the continuous parameter $R \in \,]0,1[$, where each one would have $P_R^B(z,w)$ as its reproducing kernel for which the Eq. (5.12) will represent a Zaremba expansion [22]. These RKHS are natural generalizations of the Bergman space $\mathcal{A}^B(\mathbb{D})$ and deserve to be investigated in details in a futur work.

## References


[1] M.W. Wong, Wavelet Transform and Localisation Operators, Operator Theory: Advances and Applications, Vol. 136, (2002) Birkhäuser, Basel

[2] V.V. Dodonov, 'Nonclassical' states in quantum optics: a squeezed review of the first 75 years, J.Opt. B: Quantum Semiclass. Opt. 4 (2002) R1-R33.

[3] J.P. Gazeau, Coherent states in quantum physics, WILEY-VCH Verlag GMBH & Co. KGaA Weinheim (2009)

[4] S. T. Ali, J. P. Antoine and J. P. Gazeau, Coherent States, Wavelets and Their Generalizations, second edition, Springer Science+Business Media New York (2014)

[5] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vols. I–IV ~Academic, New York (1978)

[6] I. Daubechies, Time-frequency localization operators: a geometric phase space approach, *IEEE transactions on information theory*, Vol 34, N4 (1998)

[7] D. Popov, Barut-Girardello coherent states for the pseudo-harmonic oscillator, J. Phys. A: Math. Gen. 34, 5283-5296 (2001)

[8] D. Popov, Gazeau-Klauder quasi-coherent states for the Morse oscillator, Phys. Lett. A. vol 316 (6) "69-381 (2003)

[9] Z. Mouayn, Husimi's Q-function of the isotonic oscillator in a generalized Binomial states representation, *Math. Phys. Anal. Geom.* 17(3-4):289-303, (2014)

[10] Ts Gantsog, A. Joshi and R. Tanas, Quantum Opt. vol 6 , 517-526 (1994)
[11] H-C. Fu and R. Sasaki, Negative binomial states of quantized Radiation fields, arXiv:quant-ph/9610024v1
[12] Duflo M. and Moore C.C., On the regular representation of a Nonunimodular locally compact group, *J. Funct. Anal,* **21**, 2, pp.209-243 (1976)
[13] Aslaken E W and Klauder J R , J.Math.Phys 10, p.2267 (1969).
[14] Z. Mouayn, Characterization of hyperbolic Landau states by coherent state transform, J. Phys. A: Math & Gen vol 36 (29) 8071 (2003)
[15] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the special Functions of Mathematical Physics*, 3rd edn. Die Grundlehren der mathematischen Wissenschaften, vol. 52. Springer, New York(1966)
[16] M. G. Benedict and B. Molnar, An algebraic construction of the coherent states of the Morse potential based on supersymmetric quantum mechanics, *Phys. Rev.* 60 R 1737 (1999)
[17] P. M. Morse, Diatomic molecules according to the wave mechanics.II. Vibrational levels, Phys. Rev. D. 57 (1929)
[18] M. M. Nieto and L. M. Simmons Jr: Coherent states for general potentials. I Formalism, II. Confining one-dimensional examples; III Non confining one-diemensional examples, Phys. Rev. D. Vol 20, 1321,1332,1342 (1979)
[19] H. Chhaiba, N. Demni and Z. Mouayn, Analysis of generalized negative binomial distributions attached to hyperbolic Landau levels, *J. Math. Phys*, **57**, 072103 (2016)
[20] A. Intissar, F. El Wassouli, A. Ghanmi and Z. Mouayn. Generalized second bargmann transforms associated with the hyperbolic landau levels on the poincar disk. Annales Henri Poincar, 13 (4):513-524, 2012.
[21] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev. Integrals and Series-More special functions. Volume **3**. 1986.
[22] F. H. Szafraniec, Reproducing kernel propoerty and its space : The basics , in D. Alpay (eds) Operator Theory, Springer, Basel (2015)

[1] Department of Mathematics, Faculty of Sciences and Technics (M'Ghila), Sultan Moulay Slimane University, Béni Mellal, Morocco.

[2] Institut des Hautes Études Scientifiques, Paris Saclay University Le Bois-Marie, 35 route de Chartres CS 40001 91893 Bures-sur-Yvette, France.

[3] Institut Henri Poincaré - UAR 839 Sorbonne University, / CNRS, 11 rue Pierre et Marie Curie 75231 Paris Cedex 05 France.

[4] Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden.

*Email address*: mouayn@gmail.com [1]; touhami16soumia@gmail.com [2]