Detailed examples of lift-and-project separation

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Abstract

In this document we give a detailed description of the separation of lift-and-project for two problems: stein9 and air04.

1 Two cuts of stein9

In the following we give a detailed example of the separation of two cuts for stein9. The problem objective function $\sum x_i$ has been replaced with $\sum ix_i$.

The optimal solution of the LP relaxation is $\bar{x} = (1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. We detail the procedure for generating the optimal cuts obtained from x_2 and x_5.

1.1 Notations

Through this write-up the variables are numbered using the same convention as Balas Perregaard 2003. Namely

- the structural variables of the problem are $x_1, \ldots, x_9$,
- the surplus variables for the constraints are $s_1, \ldots, s_{13}$,
- the surplus variables for the upper bounds are $s_{14}, \ldots, s_{22}$, and
- the surplus for the lower bounds (which are equal to the structurals) are $s_{23}, \ldots, s_{31}$.

The problem formulation and all tableaus are given in appendix.

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1.2 Cut generated from x_3

1.2.1 Initial basis

In the (LP) tableau The row corresponding to x_3 is given by:

$$x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 + \frac{2}{3}s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{13} - \frac{2}{3}s_{14} + \frac{2}{3}s_{30} - \frac{1}{3}s_{31} = \frac{2}{3} \quad (1)$$

The corresponding intersection cut is

$$\frac{1}{9}s_4 + \frac{1}{9}s_5 + \frac{1}{9}s_6 + \frac{2}{9}s_8 + \frac{2}{9}s_9 + \frac{4}{9}s_{13} + \frac{4}{9}s_{14} + \frac{2}{9}s_{30} + \frac{2}{9}s_{31} \geq \frac{2}{9} \quad (2)$$

In (CGLP)$_3$ To compute the objective value of the corresponding solution of $(CGLP)_3$ this cut has to be normalized to satisfy the normalization constraint $\sum u + \sum v + u_0 + v_0 = 1$. The normalization factor is $\frac{3}{16}$ and the resulting cut has violation $\frac{2}{9}\frac{3}{16} = -0.041667$.

The partition (M_1, M_2) is given by:

$$M_1 = \{s_9, s_{13}, s_{14}, s_{31}\}$$
$$M_2 = \{s_4, s_5, s_6, s_8, s_{30}\}$$

and the corresponding basis is therefore:

$$\{u_9, u_{13}, u_{14}, u_{31}, v_4, v_5, v_6, v_8, v_{30}\}$$

(We remind the reader that the basis of $(CGLP)$ is made of the variables α, u_0, v_0, u_i $\forall i \in M_1$, v_i $\forall i \in M_2$.)

Note that there is no non-basic variable with a zero coefficient in (1) therefore the basis of $(CGLP)_3$ corresponding to this basis of (LP) is unique.

1.2.2 First pivot

In the (LP) tableau A reduced cost of $-\frac{1}{4}$ is found for the variable u_{12} (corresponding to doing a negative combination with the row in which s_{12} is basic). Therefore s_{12} is selected for leaving the basis.

To choose the entering variable, we compute the function $f_{12}(\gamma)$ which gives the violation of the intersection cut generated from the combination of the source row (here the row where x_3 is basic) and γ times the row in which s_{12} is basic:

$$s_{12} + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 - 1s_{13} - 1s_{14} + 1s_{30} - 2s_{31} = 0$$

The function $f_{12}(\gamma)$ is ploted in Figure 1.

Each breakpoint in $f_{12}(\gamma)$ corresponds to a valid pivot in (LP) and the value of the function gives the objective value in the corresponding basis of $(CGLP)_3$. As can be seen from Figure 1, $f_{12}(\gamma)$ has three breakpoints:

- $\gamma = -\frac{1}{6}$ (with $f_{12}(-\frac{1}{6}) = -0.051282$),

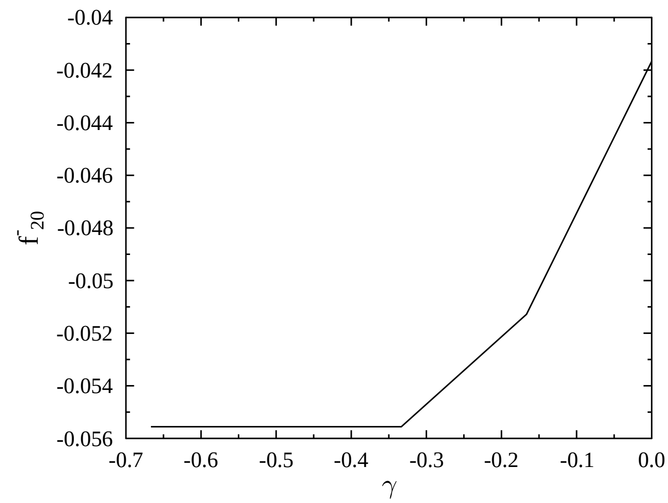

Figure 1: $f_{12}(\gamma)$ for first pivot.

- $\gamma = -\frac{1}{3}$ (with $f_{12}(-\frac{1}{3}) = -0.055556$) and
- $\gamma = -\frac{2}{3}$ (with $f_{12}(-\frac{2}{3}) = -0.05556$).

(note that all the valid pivots combining the two rows are negative). A minimum is attained for $\gamma = -\frac{2}{3}$ which corresponds to pivoting s_{14} in the basis of (LP).

Basis of $(\mathbf{CGLP})_3$ after first pivot After performing the first pivot in (LP) the basis of $(CGLP)_3$ is changed as follows

- u_{31} leaves the basis and v_{31} enters,
- v_6 leaves and u_6 enters,
- u_{14} leaves and u_{12} enters.

This gives a block pivot which could be performed in $(CGLP)_3$. This block pivot can be explained from the function $f_{12}(\gamma)$.

Note that the coefficients of the variables in the source row change monotonicaly as γ decreases. Each breakpoint corresponds to the value of γ at which a coefficient for a particular variable is zero. Let us consider a non-basic variable s_i which corresponds to a breakpoint for $\gamma = \gamma_s$ (has a zero in the source row if we take the combination of the source row with γ_i times the row of s_{12}) and

Pivot#	entering	leaving
1	u_{12}	u_{31}
2	v_{31}	v_6
3	u_6	u_{14}

Table 1: List of pivots in $(CGLP)_3$ corresponding to first pivot in (LP)

let us assume that initialy s_i is in M_1. For $\gamma > \gamma_i$ the variable stays in M_1, for $\gamma = \gamma_i$ the variable enters the basis and if $\gamma < \gamma_i$ the variable goes from M_1 to M_2.

Therefore, we can deduce the block pivot by looking at the breakpoints of the function by decreasing values (from right to left) until the minmum value of $f_{12}(\gamma)$ chosen is attained. The first breakpoint corresponds to s_{31}, the second to s_6 and the third (the one chosen) to s_{14}.

This gives the corresponding block pivot in $(CGLP)_3$ but what is the sequence of pivots which should be performed if we want each basis of $(CGLP)_3$ visited to be primal feasible?

The answer can again be deduced from $f_{12}(\gamma)$. The first pivot is to get to the first breakpoint of f_{12}, u_{12} enters the basis M_1 and u_{31} exists. When γ continues to decrease the coefficient of s_{31} in the source row becomes positive and v_{31} enters the basis of CGLP. The second breakpoint corresponds to v_{31} entering and v_6 exiting. Finaly, in the last pivot u_6 enters and u_{14} leaves the basis.

These three pivots are summarized in Table 1, it can be verified that each basis of $(CGLP)_3$ encountered during this sequence of pivots is primal feasible.

1.2.3 Second pivot

In the (LP) tableau After the first pivot, the source row reads

$$x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} + 0s_{30} + 1s_{31} = \frac{2}{3} \qquad (3)$$

The corresponding solution of $(\text{CGLP})_3$ has objective value -0.05556.

A negative reduced cost is found for the variable corresponding to the row basic in variable s_7. Minimizing $f_7(\gamma)$ for this row we find that the best cut is obtained if s_{31} enters the basis.

The next basis is optimal for $(CGLP)_3$. The intersection cut has violation -0.066667.

Second basis of CGLP Note that in the new source row of (LP) some non-basic variables have a zero coefficient. Therefore the basis of $(CGLP)_3$ is not uniquely defined anymore. When choosing s_7 to enter the basis, pivots are implicitly made in $(CGLP)_3$:

- the variable v_8 is replaced by u_8,

- the variable u_{13} is replaced by v_{13},
- the variable v_{30} is replaced by u_{30}.

The resulting basis is given by:

$$\{u_6, u_8, u_9, u_{12}, u_{30}, v_4, v_5, v_{13}, v_{31}\}$$

Second block pivot To minimize $f_7(\gamma)$ only one discontinuity point is encountered which corresponds to v_7 entering the basis v_{31} leaving it.

1.2.4 Summary

In total 2 pivots are performed in (LP) to obtain the optimal cut. These two pivots correspond to 7 pivots in $(CGLP)_3$ of which only 3 are degenerate. The final source row is:

$$x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + 0s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 - \frac{1}{3}s_9 - \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 0s_{30} = \frac{2}{3}.$$

the corresponding intersection cut is:

$$\frac{1}{9}s_4 + \frac{1}{9}s_5 + \frac{1}{9}s_7 + \frac{1}{9}s_8 + \frac{2}{9}s_9 + \frac{2}{9}s_{12} + \frac{2}{9}s_{13} \geq \frac{2}{9}.$$

The cut violation is $\frac{2}{9}$ and the normalization factor $\frac{10}{3}$, which give a optimal value of $\frac{1}{15} = 0.066667$ for $(CGLP)_3$.

1.3 Cut generated on x_6

The optimal cut from x_6 is obtained with only one pivot in the LP tableau. The source row is

$$x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{2}{3}s_6 + \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} + \frac{1}{3}s_{30} - \frac{2}{3}s_{31} = \frac{1}{3} \quad (4)$$

The objective value at the basis of $(CGLP)_6$ corresponding to the optimal basis of (LP) 0.047619. A negative reduced cost for $(CGLP)_6$ is found for the variable corresponding to the row in which s_{12} is basic.

Minimizing $f_{12}(\gamma)$ the following pivots are performed

- v_{30} leaves the basis and u_{30} enters,
- u_{31} leaves and v_{31} enters,
- u_{14} exits the basis and u_{12} enters.

The resulting cut has violation -0.083333 and is optimal for $(CGLP)_6$.

Pivot#	entering	leaving
1	u_{12}	v_{30}
2	u_{30}	u_{31}
3	v_{31}	u_{14}

Table 2: List of pivots in $(CGLP)_6$

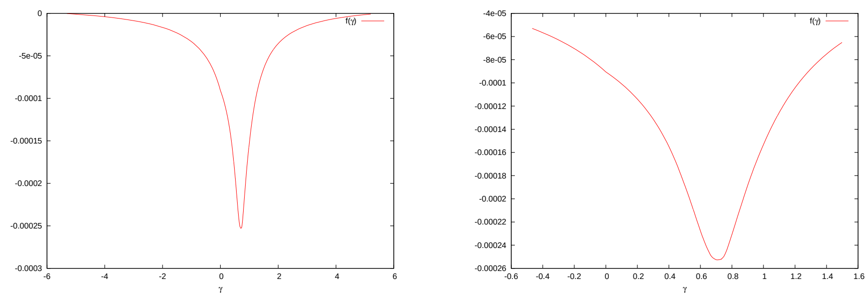

Figure 2: Plot of $f(\gamma)$ for the first pivot to generate the first cut for air04. The left figure is a plot of the function for all possible γ. The right figure plots the function for $\gamma \in [-\frac{1}{2}, \frac{3}{2}]$

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2 One cut of air04

As seen on the two cuts derived for stein9 one pivots in (LP) can correspond to several non-degenerate pivots in (CGLP). In the following we study (in less details) the separation of one cut for a large size problem: air04. The problem has 8904 variables (all binaries) and 823 constraints.

We study the separation of the lift-and-project cut corresponding to variable x_{895} for the initial linear relaxation (the cut is separated in the reduced space).

The violation of the lift-and-project cut corresponding to the initial tableau is -0.000091, 20 pivots are performed in (LP) to obtain a cut with violation -0.001317 (the procedure stops because the limit of 20 pivots is attained).

The function $f(\gamma)$ for the first pivot is plotted in Figure 2. The function has 579 discontinuity points each corresponding to a pivot. Its minimum is attained for $\gamma = 0.706161$. This pivot in (LP) corresponds to 194 pivots in $(CGLP)$.

Table 3 gives the number of pivots in $(CGLP)$ corresponding to the 20 pivots performed in (LP). In total the 20 pivots in (LP) correpsond to 1397 pivots in (CGLP).

Pivot #	cut violation	#pivots in (CGLP)
1	-0.000253	194
2	-0.000560	206
3	-0.000609	82
4	-0.000689	114
5	-0.000744	78
6	-0.000833	101
7	-0.000880	61
8	-0.000901	18
9	-0.000957	66
10	-0.000993	48
11	-0.001100	82
12	-0.001121	39
13	-0.001139	34
14	-0.001178	58
15	-0.001211	40
16	-0.001236	37
17	-0.001266	41
18	-0.001285	37
19	-0.001286	17
20	-0.001317	44

Table 3: The 20 pivots performed to obtain the lift-and-project of air04

References

[1] E. Balas and M. Perregaard. A precise correspondence between lift-and-project cuts, simple disjunctive cuts, and mixed integer Gomory cuts for 0-1 programming. *Math. Program.*, 94(2-3, Ser. B):221–245, 2003. The Aussois 2000 Workshop in Combinatorial Optimization.

A Stein9 formulation

$$\min \sum_{i=1}^{9} x_i$$

Subject To

$$x_2 + x_3 + x_4 \geq 1$$
$$x_1 + x_3 + x_5 \geq 1$$
$$x_1 + x_2 + x_6 \geq 1$$
$$x_5 + x_6 + x_7 \geq 1$$
$$x_4 + x_6 + x_8 \geq 1$$
$$x_4 + x_5 + x_9 \geq 1$$
$$x_1 + x_8 + x_9 \geq 1$$
$$x_2 + x_7 + x_9 \geq 1$$
$$x_3 + x_7 + x_8 \geq 1$$
$$x_1 + x_4 + x_7 \geq 1$$
$$x_2 + x_5 + x_8 \geq 1$$
$$x_3 + x_6 + x_9 \geq 1$$
$$\sum_{i=1}^{9} x_i \geq 4$$
$$x \in \{0, 1\}^9$$

B Successive tableaus

B.1 Optimal tableau

The optimal basis is

$$B = \{s_1, s_2, s_3, x_7, x_5, x_4, s_7, x_2, x_3, s_{10}, s_{11}, s_{12}, x_6\}$$

(this is not totaly accurate, actualy all x variables are basic by hypothesis in BP03, the basic variables are the surplus to lower bounds corresponding to these.)

The optimal tableau is (the first variable is always the basic variable in the row).

$$
\begin{aligned}
s_1 + 1s_4 + 0s_5 + 0s_6 + 0s_8 + 0s_9 - 1s_{13} - 1s_{14} + 1s_{30} + 1s_{31} &= 1 \\
s_2 + 0s_4 + 1s_5 + 0s_6 + 1s_8 + 0s_9 - 1s_{13} + 0s_{14} + 0s_{30} + 0s_{31} &= 1 \\
s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_8 + 1s_9 - 1s_{13} + 0s_{14} + 0s_{30} + 0s_{31} &= 1 \\
x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 - \frac{1}{3}s_6 - \frac{2}{3}s_8 - \frac{2}{3}s_9 + \frac{2}{3}s_{13} + \frac{2}{3}s_{14} + \frac{1}{3}s_{30} + \frac{1}{3}s_{31} &= \frac{1}{3} \\
x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{1}{3}s_6 + \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} - \frac{2}{3}s_{30} + \frac{1}{3}s_{31} &= \frac{1}{3} \\
x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{2}{3}s_6 - \frac{1}{3}s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{13} + \frac{1}{3}s_{14} + \frac{2}{3}s_{30} + \frac{2}{3}s_{31} &= \frac{2}{3} \\
s_7 + 0s_4 + 0s_5 + 0s_6 + 0s_8 + 0s_9 + 0s_{13} + 1s_{14} - 1s_{30} - 1s_{31} &= -0 \\
x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 - \frac{1}{3}s_8 + \frac{2}{3}s_9 - \frac{2}{3}s_{13} - \frac{2}{3}s_{14} - \frac{1}{3}s_{30} + \frac{2}{3}s_{31} &= \frac{2}{3} \\
x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 + \frac{2}{3}s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{13} - \frac{2}{3}s_{14} + \frac{2}{3}s_{30} - \frac{1}{3}s_{31} &= \frac{2}{3} \\
s_{10} + 0s_4 - 1s_5 - 1s_6 - 1s_8 - 1s_9 + 1s_{13} + 2s_{14} + 1s_{30} + 1s_{31} &= 1 \\
s_{11} + 0s_4 + 1s_5 + 0s_6 + 0s_8 + 1s_9 - 1s_{13} - 1s_{14} - 2s_{30} + 1s_{31} &= -0 \\
s_{12} + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 - 1s_{13} - 1s_{14} + 1s_{30} - 2s_{31} &= -0 \\
x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{2}{3}s_6 + \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} + \frac{1}{3}s_{30} - \frac{2}{3}s_{31} &= \frac{1}{3}
\end{aligned}
$$

B.2 After the first pivot for $(CGLP)_3$

$$
\begin{aligned}
s_1 + 1s_4 + 0s_5 - 1s_6 - 1s_8 + 0s_9 - 1s_{12} + 0s_{13} + 0s_{30} + 3s_{31} &= 1 \\
s_2 + 0s_4 + 1s_5 + 0s_6 + 1s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} &= 1 \\
s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} &= 1 \\
x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_8 - \frac{2}{3}s_9 + \frac{2}{3}s_{12} + 0s_{13} + 1s_{30} - 1s_{31} &= \frac{1}{3} \\
x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{2}{3}s_6 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} - 1s_{30} + 1s_{31} &= \frac{1}{3} \\
x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{12} + 0s_{13} + 1s_{30} + 0s_{31} &= \frac{2}{3} \\
s_7 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 0s_{30} - 3s_{31} &= -0 \\
x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 - 1s_8 + \frac{2}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} - 1s_{30} + 2s_{31} &= \frac{2}{3} \\
x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} + 0s_{30} + 1s_{31} &= \frac{2}{3} \\
s_{10} + 0s_4 - 1s_5 + 1s_6 + 1s_8 - 1s_9 + 2s_{12} - 1s_{13} + 3s_{30} - 3s_{31} &= 1 \\
s_{11} + 0s_4 + 1s_5 - 1s_6 - 1s_8 + 1s_9 - 1s_{12} + 0s_{13} - 3s_{30} + 3s_{31} &= -0 \\
x_1 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 1s_{30} - 2s_{31} &= 1 \\
x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} + 0s_{30} + 0s_{31} &= \frac{1}{3}
\end{aligned}
$$

B.3 After second pivot of $(CGLP)_3$

$$
\begin{aligned}
s_1 + 1s_4 + 0s_5 + 0s_6 + 1s_7 + 0s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} &= 1\\
s_2 + 0s_4 + 1s_5 + 0s_6 + 0s_7 + 1s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} &= 1\\
s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_7 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} + 0s_{30} &= 1\\
x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + 0s_6 - \frac{1}{3}s_7 - \frac{1}{3}s_8 - \frac{2}{3}s_9 + \frac{1}{3}s_{12} + \frac{1}{3}s_{13} + 1s_{30} &= \frac{1}{3}\\
x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{1}{3}s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 + \frac{1}{3}s_9 + 0s_{12} - \frac{1}{3}s_{13} - 1s_{30} &= \frac{1}{3}\\
x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{1}{3}s_6 + 0s_7 + 0s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{12} + 0s_{13} + 1s_{30} &= \frac{2}{3}\\
x_9 + 0s_4 + 0s_5 - \frac{1}{3}s_6 - \frac{1}{3}s_7 - \frac{1}{3}s_8 + 0s_9 - \frac{1}{3}s_{12} + \frac{1}{3}s_{13} + 0s_{30} &= 0\\
x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 + \frac{2}{3}s_7 - \frac{1}{3}s_8 + \frac{2}{3}s_9 + 0s_{12} - \frac{2}{3}s_{13} - 1s_{30} &= \frac{2}{3}\\
x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + 0s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 - \frac{1}{3}s_9 - \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 0s_{30} &= \frac{2}{3}\\
s_{10} + 0s_4 - 1s_5 + 0s_6 - 1s_7 + 0s_8 - 1s_9 + 1s_{12} + 0s_{13} + 3s_{30} &= 1\\
s_{11} + 0s_4 + 1s_5 + 0s_6 + 1s_7 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} - 3s_{30} &= -0\\
x_1 + 0s_4 + 0s_5 + \frac{1}{3}s_6 - \frac{2}{3}s_7 + \frac{1}{3}s_8 + 0s_9 + \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 1s_{30} &= 1\\
x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_7 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} + 0s_{30} &= \frac{1}{3}
\end{aligned}
$$

B.4 After first pivot of $(CGLP)_6$

$$s_1 + 1s_4 + 0s_5 - 1s_6 - 1s_8 + 0s_9 - 1s_{12} + 0s_{13} + 0s_{30} + 3s_{31} = 1$$
$$s_2 + 0s_4 + 1s_5 + 0s_6 + 1s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} = 1$$
$$s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} = 1$$
$$x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_8 - \frac{2}{3}s_9 + \frac{2}{3}s_{12} + 0s_{13} + 1s_{30} - 1s_{31} = \frac{1}{3}$$
$$x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{2}{3}s_6 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} - 1s_{30} + 1s_{31} = \frac{1}{3}$$
$$x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{12} + 0s_{13} + 1s_{30} + 0s_{31} = \frac{2}{3}$$
$$s_7 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 0s_{30} - 3s_{31} = -0$$
$$x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 - 1s_8 + \frac{2}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} - 1s_{30} + 2s_{31} = \frac{2}{3}$$
$$x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} + 0s_{30} + 1s_{31} = \frac{2}{3}$$
$$s_{10} + 0s_4 - 1s_5 + 1s_6 + 1s_8 - 1s_9 + 2s_{12} - 1s_{13} + 3s_{30} - 3s_{31} = 1$$
$$s_{11} + 0s_4 + 1s_5 - 1s_6 - 1s_8 + 1s_9 - 1s_{12} + 0s_{13} - 3s_{30} + 3s_{31} = -0$$
$$x_1 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 1s_{30} - 2s_{31} = 1$$
$$x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} + 0s_{30} + 0s_{31} = \frac{1}{3}$$